

For Reference

NOT TO BE TAKEN FROM THIS ROOM

Ex LIBRIS
UNIVERSITATIS
ALBERTAENSIS



For Reference

NOT TO BE TAKEN FROM THIS ROOM



Digitized by the Internet Archive
in 2020 with funding from
University of Alberta Libraries

<https://archive.org/details/Wojtiw1968>

THE UNIVERSITY OF ALBERTA

CONTINUOUS PIECEWISE INTERPOLATION

by

Lubomir Wojtiw



A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF COMPUTING SCIENCE

EDMONTON, ALBERTA

July, 1968

Thesis
1968 (F)
242

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled CONTINUOUS PIECEWISE INTERPOLATION submitted by Lubomir Wojtiw in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis presents a general development of continuous piecewise interpolation formulae extended in two directions; first the extension requiring continuity of higher order derivatives; and secondly, the extension in the multistep direction. An algorithm and a general program for the computation of the coefficient matrices of the continuous piecewise interpolation formulae in floating-point form are also given. These formulae are tested and compared on known functions.

ACKNOWLEDGEMENTS

The author acknowledges with gratitude his indebtedness to Dr. S. Charmonman for suggesting the problem explored in this thesis and for his advice, criticism and guidance throughout the period during which this research was carried out.

My sincere thanks are also due to Dr. J. Penny for his time spent in reading and offering suggestions to improve the final draft of this work and to Miss Lynne Fiveland for her time spent in typing the draft and master.

I also wish to thank Dr. D.B. Scott, Head of the Department of Computing Science, for providing computing facilities while this research was being done, and the National Research Council of Canada for financial assistance provided under research grant NRC A-4076.

TABLE OF CONTENTS

Page

CHAPTER I - INTRODUCTION

1.1	History	2
1.2	Purpose of the Study	4

CHAPTER II - BASIC THEORY

2.1	Polynomial Interpolation	5
2.1.1	Notation and Basic Definition	6
2.1.2	The Weierstrass Approximation Theorem	6
2.2	Hermite Interpolation Polynomial	7
2.2.1	Uniqueness of the Hermitian System	8
2.3	Lagrangian Interpolation Polynomial	9
2.3.1	Uniqueness of the Lagrangian System	12

CHAPTER III - ALGEBRAIC DEVELOPMENT

3.1	Preliminary	14
3.1.1	Use of Symmetry	17
3.2	Lagrangian-Hermitian Approach	17
3.2.1	Sample Derivation	17
3.2.2	General Derivation	26
3.3	Direct Derivation by Use of Polynomial Expressions	29
3.3.1	Sample Derivation	33
3.4	The Computer Algorithm	35
3.5	Discussion of Error in Machine Derivation	37
3.5.1	Calculation of Matrix D	39
3.5.2	Calculation of Matrix $F = B^{-1}D$	43
3.6	Accuracy of Machine Derivation	44

CHAPTER IV - CONTINUOUS PIECEWISE INTERPOLATION FORMULAE

4.1	Continuous First Derivative	47
4.2	Continuous First and Second Derivatives	51
4.3	Continuous First Three Derivatives	52
4.4	Generation of Desk-Calculator Tables	53

CHAPTER V - TEST RESULTS AND CONCLUSIONS

5.1 Test Functions and Procedure 55

5.2 Conclusions 58

5.3 Suggestions for Future Research 60

BIBLIOGRAPHY 61

APPENDIX I - APL\360 FUNCTIONS 66

APPENDIX II - COEFFICIENT MATRICES OF THE FORMULAE 71

Table II.1 Formulae of the 1st kind 72

Table II.2 Formulae of the 2nd kind 78

Table II.3 Formulae of the 3rd kind 82

APPENDIX III - VALUES AND DESK-COEFFICIENTS 84

Table III.1 Summary of the Formulae 85

Table III.2 Values of β and $\|\alpha F - G\|_E$ 87

Table III.3 Tables of Coefficients for Formula 2.52 89

APPENDIX IV - NUMERICAL RESULTS OF THE TEST FUNCTIONS
AND PROCEDURE 93

Table IV.1 Number of Multiplications Required 94

Table IV.2 Largest Relative Error for
 x^i , $i = 6, 7, 10$ 96

Table IV.3 Largest Relative Error for e^x 98

LIST OF FIGURES

Figure	Page
3.1 A Set of Six Points	18
3.2 Flow Diagram for Algorithm 3.1 for the Computation of the Coefficient Matrix F of (3.53) in Floating- Point	38

CHAPTER I

INTRODUCTION

Interpolation has been described as the art of reading between the lines of a table. In elementary mathematics the term usually refers to the process of computing intermediate values of a function, such as trigonometric and logarithmic, from a set of tabular values of that function. In engineering the function is usually unknown but is given in terms of a table of values obtained, for example, from measurements of physical quantities such as stream flow, precipitation, etc.. For convenience the measurements may be made at equal intervals of an independent variable. According to Weierstrass's theorem the unknown function can be approximated by a polynomial to any desired degree of accuracy provided that it is continuous over a finite interval of interest and sufficient data is available. However, in some engineering problems the use of a single polynomial for the whole set of data points may be impractical. One reason is that when the data set is very large the interpolation process may be too time-consuming if not also inaccurate due to round-off error. Another reason is that the data set may be open-ended, e.g., data is collected continually every day. Therefore in some engineering problems it is desirable or even necessary that interpolation be applied to a subset of data points at a time. The process may thus be called "piecewise interpolation".

The derivative of the function in engineering problems usually represents a physical quantity. Therefore, it is undesirable to have two values of the derivative at the joining point between any two interpolating polynomials. The process of piecewise interpolation with continuous derivatives may thus be called "continuous piecewise interpolation".

1.1 History

The theorem of Weierstrass was first proven in 1885 and since then a vast amount of literature on classical interpolation has been written. Excellent discussions of the basic concepts of interpolation may be found in Hilderbrand [17], Kopal [21] and Scarborough [25]. The discussions on classical interpolation as found in these and other standard texts on numerical analysis deal with concepts of "piecewise interpolation" without any reference to the process of "continuous piecewise interpolation". An interpolation process closely related to continuous piecewise interpolation is the well known spline interpolation.

The theory of spline approximation first appeared in a paper by Schoenberg [26]. A spline function of degree $(m - 1)$, having the knots at $x = 1, 2, \dots, (n - 1)$, is a function of the class $C^{(m-2)}(-\infty, \infty)$ which reduces to a polynomial of degree at most $(m - 1)$ in each of the intervals $(-\infty, 1)$, $(1, 2)$, \dots , $(n - 1, \infty)$. A monospline $K(x)$, of degree m , may be obtained if we add the monomial $x^m/m!$ to a spline

function. Much of the present-day theory of splines as based on Holloday's Theorem and its proof [18], requires that the second derivative at the junction point between the two interpolation polynomials be equal to zero. This will not guarantee that the function will be continuous at the junction point.

After 1946, Schoenberg, together with some of his students, continued investigations of splines and monosplines. In particular, Schoenberg and Whitney, [27] and [28], first obtained criteria for the existence of certain splines of interpolation. For the case of splines of even order with interpolation at the junction points a simpler approach to the question of existence was given by Ahlberg, Nilson, and Walsh, [4] and [6]. The convergence of the spline approximations has also come under close scrutiny. A detailed analysis was presented by Ahlberg and Nilson in [1], [2] and [3], with a modification of the restriction by Sharma and Meir in [29] and [30]. The theory of splines has also been extended in a number of directions, in particular the extension to several dimensions by de Boor, [12] and [13], and later by Ahlberg, Nilson, and Walsh in [5] and [7].

Even though many papers have been written, since the introduction of spline approximation theory in 1946, no reference is found to the process of "continuous piecewise interpolation" as introduced in 1961 by Snyder [32] for a polynomial of degree up to two with continuous first order derivatives; and extended by the same author in 1967 [34] to polynomials of degree up to four with continuous first and second order derivatives.

1.2 Purpose of the Study

Both of Snyder's formulae may be classified as "single-step" since the interpolating polynomial is moved to the right one data point at a time. In this paper we present a general development for formulae extended in two directions: firstly, an extension to higher order derivatives; and secondly, an extension in the multistep direction. Also presented is a general computer program for the computation of the coefficients of the matrix equation defining the various interpolation formulae.

CHAPTER II

BASIC THEORY

2.1 Polynomial Interpolation

Let $f(x)$ be a function which may or may not be known in closed form but the values of $f^{(j)}(x_i)$, where the superscript j indicates the j^{th} derivative, are given at the $(n + 1)$ points, i.e.

$$(2.1) \quad \begin{aligned} f^{(j)}(x_i) &= y_{ij} & i &= 0, 1, \dots, n \\ & & j &= 0, 1, \dots, (\alpha_i - 1) \end{aligned}$$

for some integers α_i and reals y_{ij} . The problem of interpolation is to approximate $f(x)$ by another function $\phi(x)$ constructed such that

$$(2.2) \quad \begin{aligned} \phi^{(j)}(x_i) &= y_{ij} & i &= 0, 1, \dots, n \\ & & j &= 0, 1, \dots, (\alpha_i - 1) \end{aligned}$$

The interpolating function $\phi(x)$ may take a variety of forms such as a polynomial, an exponential function, a trigonometric function, a continued-fraction expansion, etc.. Since a polynomial can be easily evaluated on a digital computer it will be the only form used in this thesis. A polynomial of degree n is defined as

$$(2.3) \quad P_n(x) = \sum_{i=0}^n a_i x^i$$

where n is a positive integer and a_i for $i = 0, 1, \dots, n$ are constants such that a_n is non-zero.

2.1.1 Notation and Basic Definition Upper-case Latin letters designate matrices while all lower-case letters specify scalars. A rectangular matrix $A = (a_{ij})$ with m rows and n columns is said to be of dimension m -by- n . If $m = n$, the matrix A is square and of order n . A^{-1} designates the inverse of a nonsingular matrix A and $|A|$ the determinant of A . Derivatives are specified by a superscript in parenthesis, i.e., $f^{(k)}(x)$ is the k^{th} derivative of $f(x)$. The largest integer $\leq \ell$ will be denoted by $[\ell]$.

2.1.2 The Weierstrass Approximation Theorem The justification for replacing a given function by a polynomial representation rests on the theorem stated by Weierstrass in 1885.

Theorem 2.1 *If $f(x)$ is continuous on a finite interval $[a, b]$, then, given any $\epsilon > 0$, there exists an $n = n(\epsilon)$ and a polynomial $P_n(x)$ of degree n such that $|f(x) - P_n(x)| < \epsilon$ for all x in $[a, b]$.*

The proof by Bernstein of this theorem may be found in Ralston [24]. This theorem justifies the use of polynomial approximations since it guarantees that a polynomial can be found with an arbitrary small maximum deviation from $f(x)$ on $[a, b]$.

2.2 Hermite Interpolation Polynomial

Suppose there is a basic system of interpolation functions $\phi_0(x), \phi_1(x), \dots, \phi_n(x), \dots$ on $[a, b]$. It is required to find a linear combination

(2.4)
$$\phi(x) = \sum_{i=0}^m c_i \phi_i(x) ,$$

such that (2.2) is satisfied. Since $\sum_{i=0}^n \alpha_i$ conditions have been imposed on $\phi(x)$ the problem will always have a unique solution if

(2.5)
$$m = \left(\sum_{i=0}^n \alpha_i \right) - 1$$

and

(2.6)
$$\begin{vmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_m(x_0) \\ \phi'_0(x_0) & \phi'_1(x_0) & \dots & \phi'_m(x_0) \\ \dots & \dots & \dots & \dots \\ \phi_0^{(\alpha_0-1)}(x_0) & \phi_1^{(\alpha_0-1)}(x_0) & \dots & \phi_m^{(\alpha_0-1)}(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_m(x_1) \\ \dots & \dots & \dots & \dots \\ \phi_0^{(\alpha_n-1)}(x_n) & \phi_1^{(\alpha_n-1)}(x_n) & \dots & \phi_m^{(\alpha_n-1)}(x_n) \end{vmatrix} \neq 0$$

Thus it is possible to construct an algebraic polynomial $H_m(x)$ of degree m or less which satisfies the specified conditions. This algebraic polynomial may be referred to as the Hermite interpolation polynomial [9].

2.2.1 Uniqueness of the Hermitian System We first introduce the Alternative Theorem [11], the application of which is sufficient to show that an interpolation problem has a solution, then we apply this theorem to the Hermitian system to show uniqueness.

Theorem 2.2 The homogeneous system $\sum_{j=0}^n a_{ij} x_j = 0$ $i = 0, 1, \dots, n$ possesses a non-trivial solution (i.e. a solution other than $x_i = 0$ for all $i = 0, 1, \dots, n$) if and only if $|A| = 0$. If for a given $A = (a_{ij})$ there are solutions to the non-homogeneous system, i.e., $\sum_{j=0}^n a_{ij} x_j = b_i$ $i = 0, 1, \dots, n$, for every selection of the quantities b_i , then $|A| \neq 0$ and the homogeneous system has only the trivial solution.

We now show that if $P(x)$ is a polynomial of degree N and satisfies

$$(2.7) \quad P^{(j)}(x_i) = 0 \quad \text{for } i = 0, 1, \dots, n \\ \text{and } j = 0, 1, \dots, (m_i - 1)$$

where $N = \left(\sum_{i=0}^n m_i \right) - 1$, then $P(x)$ must vanish identically. By the Factorization Theorem [11],

Theorem 2.3 If $P_n(x)$ is a polynomial of degree n then we may find n complex numbers x_1, x_2, \dots, x_n such that $a_0 \neq 0$ and

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n \equiv a_0 \prod_{i=1}^n (x - x_i) .$$

If $P(x)$ satisfies all conditions of (2.7) with the exception of the last, i.e.,

$$P^{(m_n-1)}(x_n) = 0 ,$$

then we must have

$$P(x) = A(x)(x - x_0)^{m_0} (x - x_1)^{m_1} \dots (x - x_{n-1})^{m_{n-1}} (x - x_n)^{m_n-1} ,$$

where $A(x)$ is a polynomial. By examining the degree of this product, it appears that A is a constant. Since, moreover,

$$P^{(m_n-1)}(x_n) = A(m_n - 1)! (x_n - x_0)^{m_0} \dots (x_n - x_{n-1})^{m_{n-1}} = 0$$

and since $x_i \neq x_j$, $i \neq j$, we have $A = 0$ and therefore $P(x) = 0$. The homogeneous interpolation problem has the zero solution only and hence the nonhomogeneous problem possesses a unique solution.

2.3 Lagrangian Interpolation Polynomial

This polynomial is a special case of (2.2) with $j = 0$. Without any ambiguity the superscript 0 on ϕ and the second subscript 0

on y in (2.2) may be deleted. Thus the Lagrangian interpolation problem may be defined by

$$(2.8) \quad \phi(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n$$

Consider the following polynomial of degree n ,

$$(2.9) \quad \ell_k(x) = \frac{\prod_{\substack{i=0 \\ i \neq k}}^n (x - x_i)}{\prod_{\substack{i=0 \\ i \neq k}}^n (x_k - x_i)} \quad \text{for } k = 0, 1, \dots, n$$

It is obvious that

$$(2.10) \quad \ell_k(x_j) = \delta_{kj} = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}$$

where δ_{kj} may be referred to as the Kronecker delta [11]. Therefore (2.8) will be satisfied if

$$(2.11) \quad \phi(x) = L_n(x) = \sum_{k=0}^n y_k \ell_k(x)$$

The polynomial $L_n(x)$ may be called the Lagrangian interpolation polynomial.

In some applications of interpolation, the tabular points may be equally spaced and in this case (2.11) can be simplified [19].

Let $x_i - x_{i-1} = h$ for $i = 1, 2, \dots, n$. With $x = x_0 + th$ (2.9) may be written

$$\ell_k(x) = \frac{(-1)^{n-i}}{t-i} \binom{n}{i} \frac{t(t-1)\dots(t-n)}{n!}$$

and hence (2.11) becomes

$$(2.12) \quad L_n(x) = (n+1) \binom{t}{n+1} \sum_{i=0}^n \frac{(-1)^{n-i}}{t-i} \binom{n}{i} y_i$$

The Lagrangian formula (2.11) is invariant under a linear transformation. If we let $x = hu + a$ and $x_i = hu_i + a$ then (2.9) becomes

$$(2.13) \quad \ell_k(x) = \frac{\prod_{\substack{i=0 \\ i \neq k}}^n h(u - u_i)}{\prod_{\substack{i=0 \\ i \neq k}}^n h(u_k - u_i)} = \ell_k(u)$$

for which (2.11) may be written as

$$\begin{aligned} (2.14) \quad L_n(x) &= \sum_{k=0}^n y_k \ell_k(x) \\ &= \sum_{k=0}^n y_k \ell_k(u) \\ &= L_n(u) \end{aligned}$$

2.3.1 Uniqueness of the Lagrangian System The uniqueness of the solution of this system may be obtained as a result of the following theorem, the proof of which may be found in Davis [11]:

Theorem 2.4 Given $(n + 1)$ distinct points $x_i, i = 0, 1, \dots, n$, and $(n + 1)$ values $y_i, i = 0, 1, \dots, n$, then there exists a unique polynomial $P_n(x)$ of degree n for which

$$(2.15) \quad P_n(x_i) = y_i \quad i = 0, 1, \dots, n$$

The $(n + 1)$ given data points are sufficient to specify uniquely a polynomial of degree not in excess of n ; and if it is to assume the values of y_i for x equal to $x_i, i = 0, 1, \dots, n$, its coefficients a_i for $i = 0, 1, \dots, n$ must be such that

$$(2.16) \quad \sum_{i=0}^n a_i x_j^i = y_j \quad \text{for } j = 0, 1, \dots, n$$

or expressed in matrix form,

$$(2.17) \quad \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

where the determinant of the coefficient matrix is known as the Vandermonde determinant.

When x_i, y_i for $i = 0, 1, \dots, n$ are numeric, (2.17) becomes a system of $(n + 1)$ linear equations in $(n + 1)$ unknowns and can be effectively solved by Gaussian elimination with scaling, pivoting and iterative improvement if desired [24]. If all the x_i 's are distinct, the determinant of the coefficient matrix in (2.17) is nonsingular and the solution a_i is unique. Since the interpolation problem (2.15) has a unique solution, no matter which of the finite-difference interpolation formulae, e.g. Newton's, Gauss's, Bessel's, Stirling's, etc., is employed for a given set of points, the polynomial used will be the Lagrangian polynomial.

In continuous piecewise interpolation the values of y_{ij} in (2.1) are not given for $j \geq 1$. But an expression may be obtained for these quantities in terms of y_{i0} by use of the Lagrangian interpolation formula as shown in the next chapter.

CHAPTER III

ALGEBRAIC DEVELOPMENT

The method of continuous piecewise interpolation presented herein is an extension of concepts developed by Snyder in [32] and [34]. The basic technique is continuous transformation of a segment of a polynomial curve of an arbitrary chosen degree, as interpolation proceeds across many data points covering the full range of data. The transformation of the segment is achieved by first shifting the segment a specified number of points to the right and then re-evaluating the interpolation formula. The derivatives at the junction point of the two segments are equivalent and thus the process is continuous. In addition, the application of symmetry is employed in the derivation as suggested by Charmonman and Wojtiw [10].

3.1 Preliminary

Through each of the subsets of $(n + 1)$ points of the set of data it is possible to pass a polynomial of degree n . The number of points $(n + 1)$ can take on any numeric value. For convenience, however, the same number of points could be used in all the subsets considered. Instead of requiring the Lagrangian polynomial $L_n(x)$ to pass through the $(n + 1)$ points of the subset we may use the Hermitian polynomial $H_n(x)$ to pass through $(k < n)$ points for $k \geq 2$ and to satisfy additional $(n + 1 - k)$ conditions on the derivatives at the first and

last of the k points. The additional $(n + 1 - k)$ conditions on the derivatives are not given explicitly and thus must be obtained by use of Lagrangian polynomials. Let q denote the number of points passed through by the Lagrangian polynomial. The parameter q may be either odd or even, but for convenience will be taken to be odd. Let $s = \lfloor q/2 \rfloor + 1$. Then the subset of k equally spaced points passed through by the Hermitian polynomial may be defined

$$(3.1) \quad \{x_i, i = s, (s + 1), \dots, (s + k - 1) \text{ such that } x_i > x_j \text{ if } i > j\}$$

such that

$$(3.2) \quad x_{s+i} = x_{s+i-1} + h \quad i = 1, 2, \dots, (k - 1)$$

or

$$(3.3) \quad x_{s+i} = x_s + ih \quad i = 1, 2, \dots, (k - 1)$$

To avoid having a different number of derivatives specified at the first point, i.e. x_s , than at the last point, i.e. x_{s+k-1} , we will require that if $(n + 1)$ is odd then k must be odd and if $(n + 1)$ is even then k must be even. Thus in either case $(n + 1 - k)$ will be even. For simplicity of notation, we introduce

$$(3.4) \quad m = \lfloor \frac{n + 1 - k}{2} \rfloor$$

and define the order of the multistep in terms of the subset of k points i.e.,

Definition 3.1 The order p of multistep of any formula may be defined as

$$(3.5) \quad p = k - 1$$

where k is the number of points passed through by the Hermitian polynomial.

Thus we use

$$(3.6) \quad q = n + 1 - p$$

points in each of the two Lagrangian polynomials to evaluate the m derivatives specified at x_s and at x_{s+k-1} . Therefore the Hermitian polynomial is required to satisfy

$$(3.7) \quad \phi(x_{s+i}) = y_{s+i} \quad i = 0, 1, \dots, (k - 1)$$

and

$$(3.8) \quad \phi^{(j)}(x_{s+i}) = y_{s+i}^{(j)} \quad \text{for } i = 0 \text{ and } (k - 1) \\ j = 1, 2, \dots, m$$

One Lagrangian polynomial is first passed through the subset

$$S_1 = \{y_i, i = 0, 1, \dots, (q - 1)\}$$

and another through subset

$$S_2 = \{y_i, i = (n - q + 1), (n - q + 2), \dots, (n + 1)\}$$

while the Hermitian polynomial passes through $x_s, x_{s+1}, \dots, x_{s+k-1}$.

3.1.1 Use of Symmetry For convenience in the derivation process as well as for efficient computation of the coefficients of the polynomial the tabular points in any subset of $(n + 1)$ points may be ordered symmetrically. If the number of points is even, the subscripts may be

$$(3.9) \quad \left(-\frac{n}{2}, -\frac{n}{2} + 1, \dots, -\frac{1}{2}, \frac{1}{2}, \dots, \frac{n}{2}\right)$$

and if the number of points is odd, the subscripts may be

$$(3.10) \quad \left(-\frac{n}{2}, -\frac{n}{2} + 1, \dots, -1, 0, 1, \dots, \frac{n}{2}\right)$$

3.2 Lagrangian-Hermitian Approach

3.2.1 Sample Derivation The use of Lagrangian and Hermitian

interpolation formulae in deriving the continuous piecewise interpolation formula will be illustrated by an example. Consider any set of six points symmetrically ordered, i.e. $y_{-2.5}, y_{-1.5}, y_{-.5}, y_{.5}, y_{1.5}, y_{2.5}$ with two subsets $S_1 = (y_i, i = -2.5 \text{ to } 1.5)$ and $S_2 = (y_i, i = -1.5 \text{ to } 2.5)$ as in Figure 3.1. In this example we require continuity of the first and second derivatives at the points $x = -.5$ and $x = .5$. Let $J_1(x)$ represent the Lagrangian polynomial passing through the points in subset S_1 and $J_2(x)$ the Lagrangian polynomial for subset S_2 .

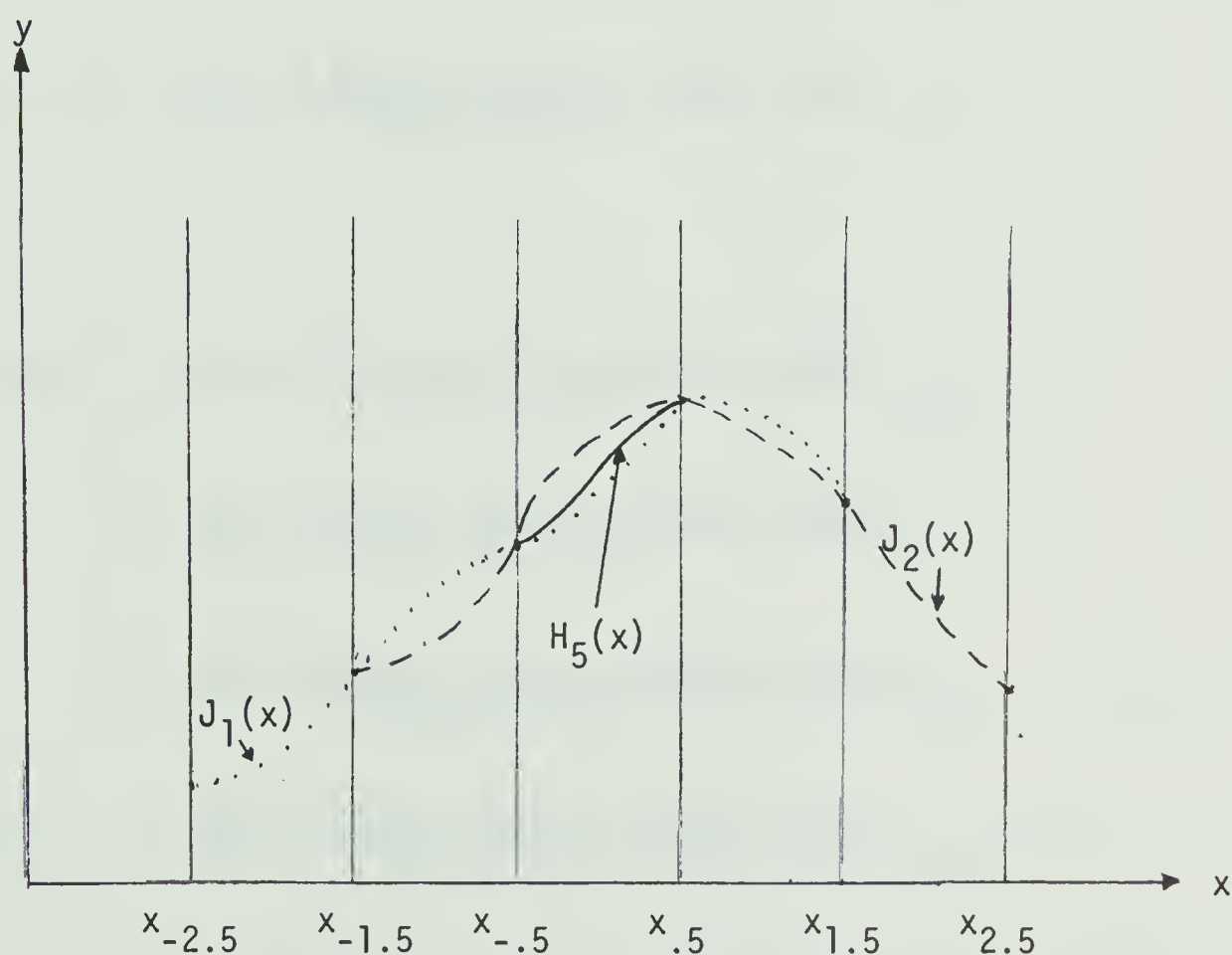


Figure 3.1 A Set of Six Points

Hence the two Lagrangian polynomials of degree four may be constructed by use of (2.9) and (2.11) with $x_i = i$ for $i = -2.5$ to 2.5 .

$$\begin{aligned}
 J_1(x) &= \frac{(x + 1.5)(x + .5)(x - .5)(x - 1.5)}{(-1)(-2)(-3)(-4)} y_{-2.5} \\
 &+ \frac{(x + 2.5)(x + .5)(x - .5)(x - 1.5)}{(1)(-1)(-2)(-3)} y_{-1.5} \\
 (3.11) \quad &+ \frac{(x + 2.5)(x + 1.5)(x - .5)(x - 1.5)}{(2)(1)(-1)(-2)} y_{-.5} \\
 &+ \frac{(x + 2.5)(x + 1.5)(x + .5)(x - 1.5)}{(3)(2)(1)(-1)} y_{.5} \\
 &+ \frac{(x + 2.5)(x + 1.5)(x + .5)(x - .5)}{(4)(3)(2)(1)} y_{1.5}
 \end{aligned}$$

and

$$\begin{aligned}
 J_2(x) &= \frac{(x + .5)(x - .5)(x - 1.5)(x - 2.5)}{(-1)(-2)(-3)(-4)} y_{-1.5} \\
 &+ \frac{(x + 1.5)(x - .5)(x - 1.5)(x - 2.5)}{(1)(-1)(-2)(-3)} y_{-.5} \\
 (3.12) \quad &+ \frac{(x + 1.5)(x + .5)(x - 1.5)(x - 2.5)}{(2)(1)(-1)(-2)} y_{.5} \\
 &+ \frac{(x + 1.5)(x + .5)(x - .5)(x - 2.5)}{(3)(2)(1)(-1)} y_{1.5} \\
 &+ \frac{(x + 1.5)(x + .5)(x - .5)(x - 1.5)}{(4)(3)(2)(1)} y_{2.5}
 \end{aligned}$$

Using (3.11) we obtain

$$(3.13) \quad J_1(-.5) = y_{-.5}$$

$$(3.14) \quad J_1'(-.5) = \frac{1}{12} (y_{-2.5} - 8y_{-1.5} + 8y_{.5} - y_{1.5})$$

and

$$(3.15) \quad J_1''(-.5) = \frac{1}{12} (-y_{-2.5} + 16y_{-1.5} - 30y_{.5} + 16y_{1.5} - y_{1.5}).$$

Similarly (3.12) gives

$$(3.16) \quad J_2(.5) = y_{.5}$$

$$(3.17) \quad J_2'(.5) = \frac{1}{12} (y_{-1.5} - 8y_{-.5} + 8y_{1.5} - y_{2.5})$$

and

$$(3.18) \quad J_2''(.5) = \frac{1}{12} (-y_{-1.5} + 16y_{-.5} - 30y_{.5} + 16y_{1.5} - y_{2.5}).$$

The second set of equations (3.16) to (3.18) may be easily obtained

from the first set (3.13) to (3.15) with a change of

indices i.e. y_i is replaced by y_{i+1} for $i = -2.5$ through to 1.5 . This

set of six equations, i.e. (3.13) through to (3.18), may be represented in matrix form as

$$(3.19) \quad \begin{bmatrix} J_1(-.5) \\ J_1'(-.5) \\ J_1^{''}(-.5) \\ J_2(.5) \\ J_2'(.5) \\ J_2^{''}(.5) \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 0 & 0 & 12 & 0 & 0 & 0 \\ 1 & -8 & 0 & 8 & -1 & 0 \\ -1 & 16 & -30 & 16 & -1 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 1 & -8 & 0 & 8 & -1 \\ 0 & -1 & 16 & -30 & 16 & -1 \end{bmatrix} \begin{bmatrix} y_{-2.5} \\ y_{-1.5} \\ y_{-.5} \\ y_{.5} \\ y_{1.5} \\ y_{2.5} \end{bmatrix}$$

We now construct a Hermitian polynomial of degree five which satisfies

$$(3.20) \quad H_5^{(j)}(-.5) = J_1^{(j)}(-.5)$$

and

$$(3.21) \quad H_5^{(j)}(.5) = J_2^{(j)}(.5)$$

where $j = 0, 1, 2$.

One way to construct $H_5(x)$ is to consider a Lagrangian interpolation polynomial $L_1(x)$ which assumes values $y_{-.5}$, $y_{.5}$ at the points $x_{-.5}$, $x_{.5}$. Therefore applying (2.9) and (2.11) $L_1(x)$ may be written as

$$(3.22) \quad L_1(x) = \frac{1}{2}(-2x + 1) y_{-.5} + \frac{1}{2}(2x + 1) y_{.5} .$$

Obviously the difference $H_5(x) - L_1(x)$ is a polynomial of degree five which will vanish at the points $x_{-.5}$ and $x_{.5}$. Thus

$$(3.23) \quad H_5(x) - L_1(x) = (x - x_{-.5})(x - x_{.5}) H_3(x)$$

where $H_3(x)$ is a Hermitian polynomial of degree three to be determined. Hence

$$(3.24) \quad H_5(x) = \frac{1}{2}(-2x + 1) y_{-.5} + \frac{1}{2}(2x + 1) y_{.5} \\ + (x + .5)(x - .5) H_3(x) .$$

Differentiating (3.24) with respect to x we have

$$(3.25) \quad H_5'(x) = -y_{-.5} + y_{.5} + (x + .5)(x - .5) H_3'(x) \\ + 2x H_3(x)$$

The condition in (3.20) with $j = 1$ is obtained by equating (3.25) with $x = -.5$ to (3.14) i.e.

$$(3.26) \quad \frac{1}{12}(y_{-2.5} - 8y_{-1.5} + 8y_{.5} - y_{1.5}) = -y_{-.5} \\ + y_{.5} - H_3(-.5) .$$

Similarly using (3.21) with $j = 1$, equation (3.25) with $x = .5$, and (3.17) yields

$$(3.27) \quad \frac{1}{12}(y_{-1.5} - 8y_{-.5} + 8y_{1.5} - y_{2.5}) = -y_{-.5} \\ + y_{.5} + H_3(.5)$$

Hence we may write

$$(3.28) \quad H_3(-.5) = \frac{1}{12}(-y_{-2.5} + 8y_{-1.5} - 12y_{-.5} + 4y_{.5} + y_{1.5})$$

and

$$(3.29) \quad H_3(.5) = \frac{1}{12}(y_{-1.5} + 4y_{-.5} - 12y_{.5} + 8y_{1.5} - y_{2.5}) .$$

Differentiating (3.25) with respect to x again, substituting $x = -.5$ and $x = .5$, and using (3.20) and (3.21) respectively we now get

$$(3.30) \quad H_3'(-.5) = \frac{1}{24}(-y_{-2.5} + 6y_{-.5} - 8y_{.5} + 3y_{1.5})$$

and

$$(3.31) \quad H_3'(.5) = \frac{1}{24}(-3y_{-1.5} + 8y_{-.5} - 6y_{.5} + y_{2.5}) .$$

Thus we have found $H_3(x)$ satisfying (3.28) through to (3.31). If we use (2.9) and (2.11) again we have

$$(3.32) \quad H_3(x) = \frac{1}{2}(-2x + 1) H_3(-.5) + \frac{1}{2}(2x + 1) H_3(.5) \\ + (x + .5)(x - .5) H_1(x)$$

where $H_1(x)$ is a Hermitian polynomial of degree one to be determined. After differentiating with respect to x and substituting $x = -.5$ and $x = .5$ we can obtain from (3.32)

$$(3.33) \quad H_1(-.5) = \frac{1}{24}(3y_{-2.5} - 14y_{-1.5} + 26y_{-.5} - 24y_{.5} \\ + 11y_{1.5} - 2y_{2.5})$$

and

$$(3.34) \quad H_1(.5) = \frac{1}{24}(-2y_{-2.5} + 11y_{-1.5} - 24y_{-.5} + 26y_{.5} \\ - 14y_{1.5} + 3y_{2.5}) .$$

Hence
$$H_1(x) = H_1(-.5) + (H_1(.5) - H_1(-.5))(x + .5)$$

and by back-substitution we obtain $H_3(x)$, which enables us to obtain $H_5(x)$ i.e.

$$\begin{aligned}
 H_5(x) = & \frac{1}{768} [x^5(-160y_{-2.5} + 800y_{-1.5} - 1600y_{-.5} + 1600y_{.5} \\
 & - 800y_{1.5} + 160y_{2.5}) + x^4(16y_{-2.5} - 48y_{-1.5} + 32y_{-.5} + 32y_{.5} \\
 & - 48y_{1.5} + 16y_{2.5}) + x^3(144y_{-2.5} - 848y_{-1.5} + 1824y_{-.5} - 1824y_{.5} \\
 (3.35) \quad & + 848y_{1.5} - 144y_{2.5}) + x^2(-40y_{-2.5} + 312y_{-1.5} - 272y_{-.5} - 272y_{.5} \\
 & + 312y_{1.5} - 40y_{2.5}) + x(-26y_{-2.5} + 162y_{-1.5} - 1124y_{-.5} \\
 & + 1124y_{.5} - 162y_{1.5} + 26y_{2.5}) + (9y_{-2.5} - 75y_{-1.5} + 450y_{-.5} \\
 & + 450y_{.5} - 75y_{1.5} + 9y_{2.5})].
 \end{aligned}$$

With a_i defined as the coefficient of the Hermitian polynomial of degree five

$$(3.36) \quad H_5(x) = \sum_{i=0}^5 a_i x^i$$

(3.35) may be expressed in matrix form

$$(3.37) \quad \frac{1}{768} \begin{bmatrix} 9 & -75 & 450 & 450 & -75 & 9 \\ -26 & 162 & -1124 & 1124 & -162 & 26 \\ -40 & 312 & -272 & -272 & 312 & -40 \\ 144 & -848 & 1824 & -1824 & 848 & -144 \\ 16 & -48 & 32 & 32 & -48 & 16 \\ -160 & 800 & -1600 & 1600 & -800 & 160 \end{bmatrix} \begin{bmatrix} y_{-2.5} \\ y_{-1.5} \\ y_{-.5} \\ y_{.5} \\ y_{1.5} \\ y_{2.5} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

3.2.2 General Derivation We now consider the general problem as stated by (3.7) and (3.8) subject to the definitions and conditions specified in section 3.1. The Lagrangian interpolation formula (2.11) may be expressed in symmetric form. For the number of points $(n + 1)$ we may write

$$(3.38) \quad L_n(x) = \sum_{i=-n/2}^{n/2} y_i \ell_i(x)$$

where

$$\ell_i(x) = \frac{\prod_{\substack{j=-n/2 \\ j \neq i}}^{n/2} (x - x_j)}{\prod_{\substack{j=-n/2 \\ j \neq i}}^{n/2} (x_i - x_j)}$$

Also if $J_1(x)$ denotes the Lagrangian polynomial $L_{q-1}(x)$ passing through the points in subset $S_1 = \{y_i, \text{ for the first } q \text{ points of } (n+1)\}$ and $J_2(x)$ for the subset $S_2 = \{y_i, \text{ for the last } q \text{ points of } (n+1)\}$ then for the subset of k points we may evaluate k equations i.e.

$$(3.39) \quad J_1(x)|_{x=x_s}, J_w(x)|_{x=x_{s+1}}, \dots, J_w(x)|_{x=x_{s+k-2}}, J_2(x)|_{x=x_{s+k-1}}$$

where w may be either 1 or 2 depending on which is more appropriate. This corresponds to the left-hand side of (3.7). We next evaluate the m derivatives of (3.38) at the two points x_s and x_{s+k-1} , by first differentiating the equation with respect to x and then evaluating the required derivative at these two points to yield m equations for the point x_s , i.e.

$$(3.40) \quad J_1'(x)|_{x=x_s}, J_1''(x)|_{x=x_s}, \dots, J_1^m(x)|_{x=x_s}$$

and an additional m equations for x_{s+k-1} , i.e.

$$(3.41) \quad J_2'(x)|_{x=x_{s+k-1}}, J_2''(x)|_{x=x_{s+k-1}}, \dots, J_2^m(x)|_{x=x_{s+k-1}}$$

A total of $2m$ equations, i.e. (3.40) and (3.41), correspond to the left-hand side of (3.8). Thus a total of $(2m + k)$ equations are formed by (3.39), (3.40) and (3.41).

We now use Lagrangian formula (3.38), to define another polynomial $L_{k-1}(x)$ where k is the number of points in the subset of k points. We further assume that there exists a polynomial of degree n , denoted by $H_n(x)$ satisfying the right-hand side conditions of equations (3.7) and (3.8). The difference $H_n(x) - L_{k-1}(x)$ will be a polynomial of degree n , which will vanish at the points x_{s+i} , $i = 0, 1, \dots, (k - 1)$. Consequently,

$$H_n(x) - L_{k-1}(x) = \omega_k(x) H_{n-k}(x)$$

where

$$(3.42) \quad \omega_k(x) = \prod_{i=-\left(\frac{k-1}{2}\right)}^{\left(\frac{k-1}{2}\right)} (x - x_i)$$

For any polynomial $H_{n-k}(x)$ the function

$$(3.43) \quad H_n(x) = L_{k-1}(x) + \omega_k(x) H_{n-k}(x)$$

will assume the values of y_i at the points of interpolation. We now obtain $H_{n-k}(x)$ such that the remaining conditions are satisfied, by first differentiating both sides of (3.43) with respect to x and evaluating $x = x_i$ where $i = s$ and $(s + k - 1)$ i.e.

$$(3.44) \quad H'_n(x_i) = L'_{k-1}(x_i) + \omega'_k(x_i) H_{n-k}(x_i) + \omega_k(x_i) H'_{n-k}(x_i)$$

but $\omega_k(x_i) H'_{n-k}(x_i) \equiv 0$ since $\omega_k(x_i) \equiv 0$. Since $\omega'_k(x_i) \neq 0$ we find $H_{n-k}(x_i)$ at each point where $H'_n(x_i)$ is defined. This process may be continued m times for the m derivatives of the right-hand side of (3.8). Each time the coefficient of the highest derivative of $H_{n-k}(x)$ at the points x_i will be $\omega'_k(x_i)$. Thus the problem of finding $H_n(x)$ becomes that of finding $H_{n-k}(x)$ satisfying the conditions

$$(3.45) \quad H_{n-k}^{(j)}(x_{s+i}) = \frac{d^j}{dx^j} \left(\frac{H_n(x) - L_{k-1}(x)}{\omega_k(x)} \right) \quad \begin{array}{l} j = 1, 2, \dots, (m-1) \\ i = 0 \text{ and } (k-1) \end{array}$$

and

$$(3.46) \quad H_{n-k}(x_{s+i}) = \frac{H_n(x) - L_{k-1}(x)}{\omega_k(x)} \quad i = 0, 1, \dots, (k-1)$$

Similarly the system (3.45) and (3.46) may be reduced. After carrying out this process m times, it is possible by back-substitution to obtain the final result in a matrix form.

3.3 Direct Derivation by Use of Polynomial Expressions

A more direct approach to obtain a continuous piecewise interpolation formula is to assume a $(v = q-1)^{\text{th}}$ degree polynomial $Q_v(x)$ passing through the q points in either subset S_1 or S_2 i.e.

$$(3.47) \quad Q_v(x) = \sum_{i=0}^v a_i x^i$$

such that $x = -g, (-g + 1), \dots, 0, 1, \dots, g$ where $g = \frac{v}{2}$ and q has been taken for convenience to be odd. Thus for subset S_2

$$(3.48) \quad Q_v(-g + i) = y^{-\frac{n}{2} + i + p} \quad i = 0, 1, \dots, v$$

while for subset S_1 (3.48) holds with $p = 0$. Equation (3.47) may be written in matrix form as

$$(3.49) \quad \begin{bmatrix} Q_v(-g) \\ Q_v(-g+1) \\ \cdot \\ \cdot \\ Q_v(0) \\ \cdot \\ \cdot \\ Q_v(g-1) \\ Q_v(g) \end{bmatrix} = \begin{bmatrix} 1 & (-g)^1 & (-g)^2 \dots (-g)^v \\ 1 & (-g+1)^1 & (-g+1)^2 \dots (-g+1)^v \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 0 & 0 \dots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & (g-1)^1 & (g-1)^2 \dots (g-1)^v \\ 1 & (g)^1 & (g)^2 \dots (g)^v \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{v-1} \\ a_v \end{bmatrix}$$

x_s for subset S_1 corresponds to $x = 0$ in (3.47). We may also calculate the inverse of the coefficient matrix in (3.49) to obtain the coefficient a_0 in (3.47) which represents the left-hand side of (3.7) in terms of y_i 's at x_s . Coefficients a_i after multiplication by the factorial of the i^{th} power of x in (3.47) yield the left-hand side of (3.8) in terms of y_i 's at x_s . Similarly we can let $x = 0$ correspond to the x_{s+k-1} point in the subset S_2 and obtain the same coefficients as for x_s , except for $y_{-\frac{n}{2}}$ now replaced by $y_{-\frac{n}{2} + p}$. Following this procedure for (3.7) and (3.8) we may obtain the system

$$(3.50) \quad c = Dy$$

where c is a vector of $(2m + k)$ conditions as given by the left-hand side of (3.7) and (3.8), D a matrix of dimension $(n + 1)$ -by- $(2m + k)$ expressed in terms of the derived coefficients, and y a vector of length $(n + 1)$ of the y_i 's. This is equivalent to passing a Lagrangian polynomial through the points in subsets S_1 and S_2 .

We now assume that another polynomial,

$$(3.51) \quad H_{2m+k-1}(x) = \sum_{i=0}^{2m+k-1} e_i x^i$$

provides the basic form for the Hermitian interpolation function. The $(2m + k)$ coefficients of this polynomial satisfy the requirements given by (3.7) and (3.8). We obtain k equations for (3.7) and $2m$ equations

for (3.8). The total of $(2m + k)$ equations may be written as

$$(3.52) \quad d = Bz$$

where d is a vector of the $(2m + k)$ conditions as given by (3.7) and (3.8), B a matrix of order $(2m + k)$ expressed in terms of the derived coefficients, and $z = \{e_0, e_1, \dots, e_{2m+k-1}\}^T$.

Since $c = d$, from (3.7) and (3.8), then (3.50) and (3.52) with $F = B^{-1}D$ becomes

$$(3.53) \quad z = Fy$$

where $z = \{e_0, e_1, \dots, e_{2m+k-1}\}^T$, F a matrix of dimension $(n + 1)$ -by- $(2m + k)$ giving numeric coefficients, and y the vector of y_i 's of length $(n + 1)$. Hence (3.53) is the matrix equation defining the various interpolation formulae and F is its coefficient matrix. Thus (3.51) can now be expressed in terms of the tabular points y_i 's. It can be noted that the coefficient matrix F will be square if

$$(3.54) \quad q = 2m+1,$$

and that a unique solution exists only if

$$(3.55) \quad 2m+k \leq n+1.$$

3.3.1 Sample Derivation If we now consider the same example as in sub-section 3.2.1, then (3.49) may be written

$$(3.56) \quad \begin{bmatrix} y_{-2.5} \\ y_{-1.5} \\ y_{-.5} \\ y_{.5} \\ y_{1.5} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

for the subset $S_1 = \{y_i, i = -2.5 \text{ to } 1.5\}$. The expression for $S_2 = \{y_i, i = -1.5 \text{ to } 2.5\}$ is exactly the same as (3.56) except for the left-hand side where y_i is replaced by y_{i+1} for $i = -2.5$ to 1.5 . The inverse of (3.56) becomes

$$(3.57) \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 0 & 0 & 24 & 0 & 0 \\ 2 & -16 & 0 & 16 & -2 \\ -1 & 16 & -30 & 16 & -1 \\ -2 & 4 & 0 & -4 & 2 \\ 1 & 4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_{-2.5} \\ y_{-1.5} \\ y_{-.5} \\ y_{.5} \\ y_{1.5} \end{bmatrix}$$

Thus

$$\begin{aligned} \phi(-.5) &= \frac{1}{24} (0y_{-2.5} + 0y_{-1.5} + 24y_{-.5} + 0y_{.5} + 0y_{1.5}) \\ &= y_{-.5} \end{aligned}$$

while

$$\begin{aligned}\phi'(-.5) &= \frac{1}{24} (2y_{-2.5} - 16y_{-1.5} + 0y_{-.5} + 16y_{.5} - 2y_{1.5}) \times 1! \\ &= \frac{1}{12} (y_{-2.5} - 8y_{-1.5} + 8y_{.5} - y_{1.5})\end{aligned}$$

and

$$\begin{aligned}\phi''(-.5) &= \frac{1}{24} (-y_{-2.5} + 16y_{-1.5} - 30y_{-.5} + 16y_{.5} - y_{1.5}) \times 2! \\ &= \frac{1}{12} (-y_{-2.5} + 16y_{-1.5} - 30y_{-.5} + 16y_{.5} - y_{1.5})\end{aligned}$$

Similarly, $\phi(.5)$, $\phi'(.5)$ and $\phi''(.5)$ may be obtained to yield the coefficient matrix

$$(3.58) \quad D = \frac{1}{12} \begin{bmatrix} -1 & 16 & -30 & 16 & -1 & 0 \\ 1 & -8 & 0 & 8 & -1 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 1 & -8 & 0 & 8 & -1 \\ 0 & -1 & 16 & -30 & 16 & -1 \end{bmatrix}$$

For the basic form of the interpolation function we assume a general fifth-degree polynomial

$$(3.59) \quad H_5(x) = \sum_{i=0}^5 e_i x^i$$

for which we may obtain the matrix form

$$(3.60) \quad \begin{bmatrix} H_5''(-.5) \\ H_5'(-.5) \\ H_5(-.5) \\ H_5(.5) \\ H_5'(.5) \\ H_5''(.5) \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 0 & 0 & 64 & -96 & 96 & -80 \\ 0 & 32 & -32 & 24 & -16 & 10 \\ 32 & -16 & 8 & -4 & 2 & -1 \\ 32 & 16 & 8 & 4 & 2 & 1 \\ 0 & 32 & 32 & 24 & 16 & 10 \\ 0 & 0 & 64 & 96 & 96 & 80 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}$$

The coefficient-matrix of (3.60) is B in (3.52). We can now calculate the inverse of B and, with D from (3.58), obtain equation (3.37).

3.4 The Computer Algorithm

A computer program may be written using the approach presented in section 3.3 to obtain the coefficient matrix F of (3.53) in floating-point representation.

Algorithm 3.1 Given the three variables k, q, and m where k is the number of points passed through by the Hermitian polynomial, q the number of points used in evaluating the derivatives, and m the highest order of the derivative required for continuity.

Step 1: Test if the selection of k, q, and m yields a unique solution.

If not, then exit.

Step 2: Calculate the coefficient matrix B in (3.52), which provides the basic form for the Hermitian interpolation function, as

follows:

Step 2.1: Set up matrix B of order $(2m + k)$ containing zeros and calculate $x = i - \frac{1}{2}(k + 1)$, $i = 1, 2, \dots, k$ i.e. the tabular points through which the Hermitian polynomial passes.

Step 2.2: Evaluate x^j in (3.51) for $j = 0, 1, \dots, (2m + k - 1)$ using x in step 2.1 and place these in the $(m + i)^{\text{th}}$ row of B. This gives the requirements of equation (3.7).

Step 2.3: Evaluate the m derivatives at $x = -\frac{1}{2}(k - 1)$ and $x = \frac{1}{2}(k - 1)$ and place these in the first m and the last m rows of B respectively. Thus the requirements of (3.8) are satisfied.

Step 3: Now calculate the coefficient matrix D in (3.50). This may be considered as passing the Lagrangian polynomial through subsets S_1 and S_2 .

Step 3.1: Set up the coefficient matrix D of dimension $(k + q - 1)$ -by- $(2m + k)$ such that $D = \{d_{ij}\}$, $j = 1, 2, \dots, (k + 2m)$ and $i = 1, 2, \dots, (k + q - 1)$ yields $d_{ij} = 1$ if $j = i - [m + \frac{1}{2}(1 - q)]$ and $d_{ij} = 0$ otherwise. This is equivalent to satisfying conditions (3.7).

Step 3.2: Calculate $x = i - \frac{1}{2}(q + 1)$ for $i = 1, 2, \dots, q$ and evaluate x^j in (3.47) for $j = 0, 1, \dots, (q - 1)$.

This now can be expressed as the coefficient matrix of (3.49).

Step 3.3: Obtain the inverse of this coefficient matrix by Gaussian elimination.

Step 3.4: Use the $(i + 1)^{\text{th}}$ rows of this inverse times $i!$ for $i = 1, 2, \dots, m$ to obtain the m derivatives of (3.8) at $x = -\frac{1}{2}(k - 1)$ and at $x = \frac{1}{2}(k - 1)$, and place these in the first m and the last m rows of D respectively. Hence the coefficient matrix D is set up.

Step 4: Calculate B^{-1} by Gaussian elimination and obtain $F = B^{-1} D$.

F is now the desired coefficient matrix in floating-point form.

A flow diagram for such a program is given in Figure 3.2.

3.5 Discussion of Error in Machine Derivation

We now investigate the computational errors arising in the derivation of a continuous piecewise interpolation formula by a digital computer. All the computations and results presented herein were performed on an IBM S/360/67 computer using APL\360 or Iverson's language [14], [20]. In APL\360 all floating-point computations are performed in extended long word (64 bits with 56 bits of fractional part) and the final result may be rounded and displayed in 1 to 17 significant figures. The results presented herein on error bounds

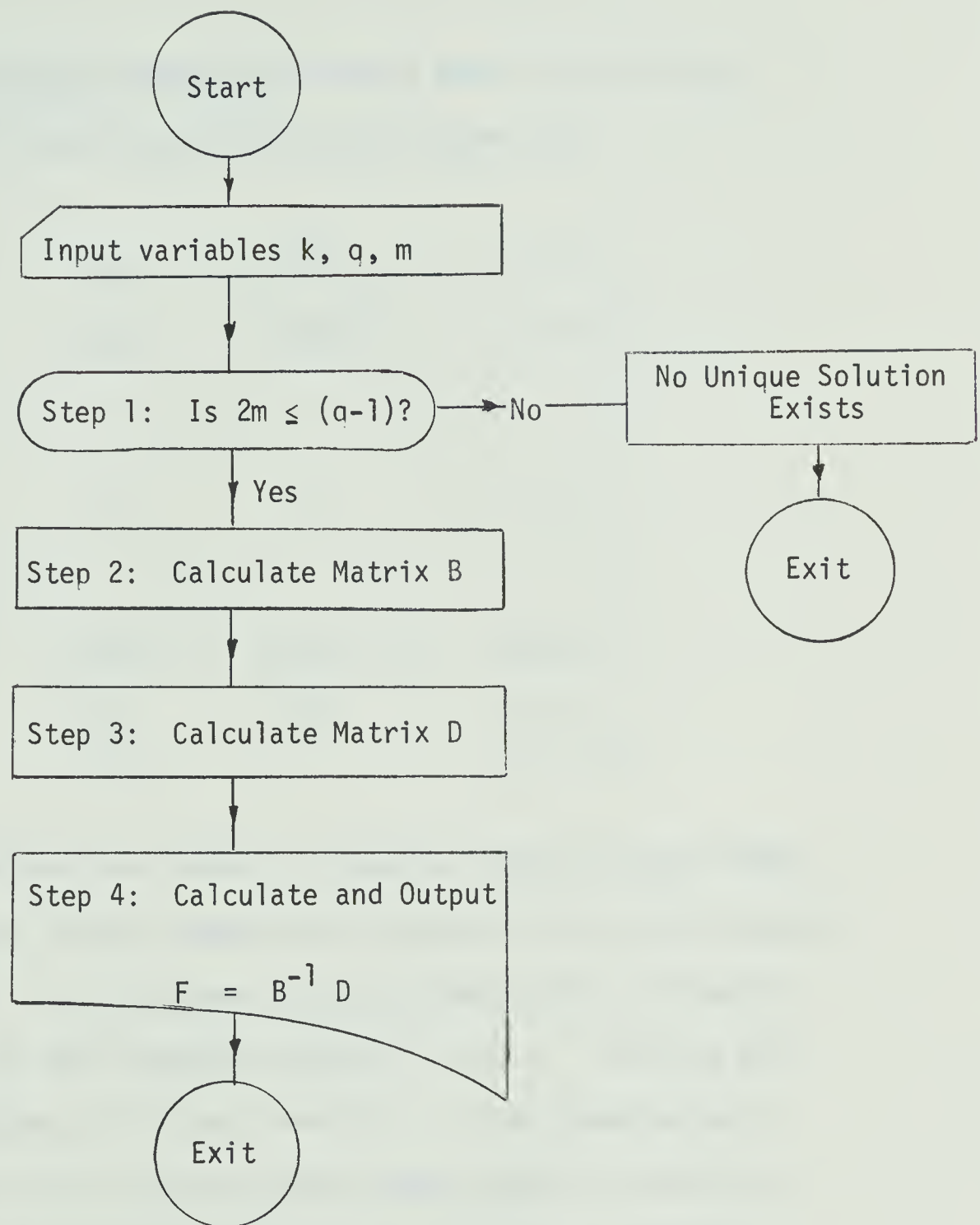


Figure 3.2 Flow diagram for Algorithm 3.1 for the computation of the coefficient matrix F of (3.53) in floating-point.

are due to Wilkinson [36] and Forsythe and Moler [15].

3.5.1 Calculation of Matrix D Since g and i in (3.47) are integers the coefficient matrix W of (3.49) (step 3.2)

$$W = \begin{bmatrix} 1 & (-g) & (-g)^2 & \dots & (-g)^v \\ 1 & (-g+1) & (-g+1)^2 & \dots & (-g+1)^v \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (g-1) & (g-1)^2 & \dots & (g-1)^v \\ 1 & (g) & (g)^2 & \dots & (g)^v \end{bmatrix}$$

may be obtained without any error if a special routine using integer arithmetic is used. We now compute the inverse of W by using Gaussian elimination [24]. In this method the scaled matrix W is decomposed into the product of two triangular matrices L and U . Pivoting only involves permutations of the row subscripts. Since permutation is irrelevant to the error analysis we may simply neglect pivoting in our analysis. The decomposition consists of computing a sequence of matrices $W^{(1)} = W, W^{(2)}, \dots, W^{(n)}$, where the matrix $W^{(k)}$ has zero elements below the diagonal in the first $(k - 1)$ columns. The

matrix $W^{(k+1)}$ is obtained from $W^{(k)}$ by subtracting a multiple of the k^{th} row from each of the rows below it and the rest of $W^{(k)}$ is left unchanged. The multipliers are chosen so that if there were no rounding errors $W^{(k+1)}$ would have zeros below the diagonal in the k^{th} column. Let $W^{(k)} = \{w_{i,j}^{(k)}\}$ and

$$(3.62) \quad m_{i,k} = \text{fl} (w_{i,k}^{(k)} / w_{k,k}^{(k)}) , \quad i \geq k + 1$$

$$(3.63) \quad w_{i,j}^{(k+1)} = \begin{cases} 0 & \text{for } i \geq k + 1, j = k \\ \text{fl} (w_{i,j}^{(k)} - m_{i,k} \times w_{k,j}^{(k)}) & \text{for } i \geq k + 1 \text{ and } j \geq k + 1 \\ w_{i,j}^{(k)} & \text{otherwise} \end{cases}$$

Also let

$$(3.64) \quad U = W^{(n)}$$

and

$$(3.65) \quad L = \begin{bmatrix} 1 & & & & \\ m_{2,1} & 1 & & & \\ m_{3,1} & m_{3,2} & 1 & & \\ \dots & \dots & \dots & \dots & \\ m_{n,1} & m_{n,2} & \dots & \dots & 1 \end{bmatrix}$$

Obviously L and U are lower and upper triangular, respectively. The following theorem is stated and proven by Forsythe and Moler [15].

Theorem 3.1 *The solution x computed by Gaussian elimination with pivoting satisfies the equation*

$$(3.66) \quad (W + \delta W)x = b \quad .$$

Furthermore

$$(3.67) \quad \|\delta W\|_{\infty} \leq 1.01(n^3 + 3n^2) \rho \|W\|_{\infty} u \quad .$$

where

$$\rho = \max_{i,j,k} |w_{i,j}^{(k)}| / \|W\|_{\infty}$$

and u the unit round-off i.e.

$$u = \frac{1}{2} \beta^{1-t} \quad (\text{rounded operations})$$

$$u = \beta^{1-t} \quad (\text{chopped operations}) \quad .$$

On the IBM S/360/67 using APL\360, the value of $\beta = 2$ and $t = 53$. For even approximate equality to hold in (3.67) the "worst" round-off must occur at each step of the calculation and, furthermore, equations like $\|AB\|_{\infty} = \|A\|_{\infty} \cdot \|B\|_{\infty}$ must hold. Forsythe and Moler state that no example is known where $\|\delta W\|_{\infty}$ is even close to the given bound. Wilkinson [36, p. 108] states that $\|\delta W\|_{\infty}$ is rarely larger than $n \|W\|_{\infty}$. He further gives the bound of δW for sufficiently large n as

$$(3.68) \quad \|\delta W\|_{\infty} \leq g_1 2^{-t_1} (2.005n^2 + n^3 + .25n^4 2^{-t_1}) \|W\|_{\infty}$$

where g_1 is the largest element of any $|W^{(k)}|$ and

$$(3.69) \quad 2^{-t_1} = (1.06) 2^{-t}$$

Also he remarks that the last term in the parenthesis of (3.68) is unimportant if $n2^{-t} \ll 1$, and the dominant term is $g_1 2^{-t_1} n^3$. The dominant term comes from errors in the solution of the triangular system of equations. The bound in (3.68) may only be attained in exceptional circumstances, so that even the replacement of the factor $2.005n^2 + n^3$ by its square root gives a somewhat higher value than would normally be expected in practice. Either the results of Wilkinson or Forsythe and Moler may be used to give a bound for the calculation of W^{-1} . Let E_1 be the error in obtaining W^{-1} . The coefficient matrix

D obtained by step 3.4 in Algorithm 3.1 of section 3.4 will not result in any additional errors. Thus the error in obtaining the matrix D is also E_1 .

3.5.2 Calculation of Matrix $F = B^{-1} D$ The matrix B in step 2 of Algorithm 3.1 of section 3.4 can also be obtained without any error. But its inverse computed by Gaussian elimination may contain an error, denoted by E_2 , for which the theorem in sub-section 3.5.1 will also hold. A final error, denoted by E_3 , may arise when the products of two matrices $B^{-1} D$ is calculated. Thus we have

$$\begin{aligned}
 (3.70) \quad f1(B^{-1} D) &\equiv B^{-1} D + E \\
 &= (B^{-1} + E_2)(D + E_1) + E_3 \\
 &= B^{-1} D + E_2 D + B^{-1} E_1 + E_3 .
 \end{aligned}$$

Thus the total error E is

$$(3.71) \quad E = E_2 D + B^{-1} E_1 + E_3$$

Let us consider $E_1 = E_2 = 0$. This is equivalent to assuming that no error arises in calculating the inverse, since it is possible to obtain the inverse of an integer matrix in integer form. This is done in hand computation by performing all calculations to the common denominator. Then (3.70) becomes

$$(3.72) \quad fl(B^{-1} D) = B^{-1} D + E_3$$

where E_3 is the error due to the product of $B^{-1} D$. For the product of two matrices in floating-point Wilkinson [36] gives the result

$$(3.73) \quad \|E_3\|_E \leq 2^{-t_1} n \|B^{-1}\|_E \|D\|_E$$

where t_1 satisfies (3.69) and the Euclidean norm for any matrix A is

$$(3.74) \quad \|A\|_E = \left(\sum_i \sum_j |a_{ij}|^2 \right)^{1/2}$$

The bounds given are hardly likely to be attained since $\|B^{-1}\|_E \|D\|_E$ may be very much greater than $\|B^{-1} D\|_E$ and the bound for E_3 may be much greater than the actual error.

3.6 Accuracy of Machine Derivation

Although no reasonable bound of round-off error can be obtained we can check the actual error of the result of the machine derivation. It is possible to follow Algorithm 3.1 given in section 3.4 using hand computation. Such computations may be carried out in integer arithmetic and expressed in terms of a common denominator. Thus the results obtained are without any round-off error at all. Comparison of the exact result to the floating-point machine derivation revealed that the actual errors in the floating-point result were so

small that it is possible to convert the floating point result into integer form by multiplying it by a certain number, α .

If α is the multiplying factor such that the floating-point representation may be replaced by an integer form then it is possible to obtain a numerical value for

$$(3.75) \quad \max_{i,j} |\alpha a_{ij} - b_{ij}| \leq \epsilon_{ij} \quad \text{all } i, j$$

and

$$(3.76) \quad \beta = \min \epsilon_{ij}$$

where $F = \{a_{ij}\}$ and $G = \{b_{ij}\}$ i.e. F is the computed coefficient matrix in floating-point while G is the exact one obtained by hand computation in integer form. The results of the cases considered for β and $\|\alpha F - G\|_E$ (see (3.74)) are given in Table III.2 of Appendix III.

CHAPTER IV

CONTINUOUS PIECEWISE INTERPOLATION FORMULAE

Various continuous piecewise interpolation formulae can be constructed either in integer or floating-point notation. The formulae depend upon three basic parameters, k , q , and m , where k is the number of points through which the Hermitian polynomial passes, q the number of points used in evaluating the derivatives, and m the highest order of the derivative required for continuity. Other quantities, i.e. total number of points, the order of the multistep etc., may be used but these may be expressed in terms of k , q , and m . Thus the various formulae presented may be classified according to the format

$$(4.1) \quad \{k, q, m\} \quad .$$

In order that a unique solution will exist the parameters are subject to the conditions given in Chapter III. The formulae presented in this chapter are for k up to five, q up to eleven and m up to three. Other formulae with higher values of the three parameters may be obtained in a similar manner.

For identification purpose any formula of the m^{th} kind is defined to be the formula which has continuity up to the m^{th} derivative. The formulae of the m^{th} kind may be sub-divided according to the order of the multistep. For classifying the order $(k - 1)$ of the

multistep, adjectives such as single, double, triple, etc. will be used to modify the noun "formula". Further modification of the form $(n + 1 = k + q - 1)$ -point will be used to indicate the total number of points used in the subset. As an example, a formula requiring continuity up to the second derivative, with the order of the multistep being one and using six points would be called the six-point single-step formula of the second kind. The results for the cases considered are summarized in Table III.1 of Appendix III.

The use of symmetry as mentioned in Subsection 3.1.1 results in the absolute value of the coefficient matrix being symmetric in a sense not the same as in matrix algebra (see Appendix II). If the total number of points used is even, symmetry is about a line between the centre most columns. For an odd number of points the symmetry is about the middle column.

Also for each of the continuous piecewise interpolation formulae it is possible to generate a set of desk-calculator tables on an electronic computer. A procedure for obtaining such tables is given in section 4.4 of this chapter. An example of such tables may be found in Table III.3 of Appendix III.

4.1 Continuous First Derivative

A number of different continuous piecewise interpolation formulae with continuous first derivative may be obtained. As mentioned before, these may be called formulae of the first kind since $m = 1$. The

simplest possible formula for odd q is when $k = 2$ and $q = 3$. Hence a total of $n + 1 = k + q - 1 = 4$ points $\{x_{-1.5}, x_{-.5}, x_{.5}, x_{1.5}\}$ are used with the first derivative at $x_{-.5}$ expressed in terms of the first three points and at $x_{.5}$ in terms of the last three points. The Hermitian polynomial satisfies four conditions and passes through the two interior points. By format (4.1) this formula may be called Formula 2.3.1 or the four-point single-step formula of the first kind. Here the interpolating polynomial is moved to the right one data point at a time accounting for the phrase "single-step". The coefficient matrix F of (3.53) in this case is

$$(4.2) \quad \frac{1}{16} \begin{bmatrix} -1 & 9 & 9 & -1 \\ 2 & -22 & 22 & -2 \\ 4 & -4 & -4 & 4 \\ -8 & 24 & -24 & 8 \end{bmatrix}$$

and is square as may be expected from (3.54). This formula is exact to a polynomial of degree two since the lowest degree of the polynomials used is two.

As an extension of Formula 2.3.1 we may consider $q = 5$ for which Formula 2.5.1 or the six-point single-step formula of the first kind, may be obtained. As in the case of Formula 2.3.1 the interpolating polynomial also moves one data point at a time to the right. Formula 2.5.1 is exact to a polynomial of degree three since the lowest degree

of the polynomials used is three. In this case the coefficient matrix F of (3.53) is rectangular,

$$(4.3) \quad \frac{1}{96} \begin{bmatrix} 1 & -9 & 56 & 56 & -9 & 1 \\ -2 & 14 & -128 & 128 & -14 & 2 \\ -4 & 36 & -32 & -32 & 36 & -4 \\ 8 & -56 & 128 & -128 & 56 & -8 \end{bmatrix}$$

Similarly we may obtain coefficient matrices for $q = 7, 9$ and 11 and denote these as Formulae 2.7.1, 2.9.1, and 2.11.1 respectively. All these formulae are single-step and exact only to a polynomial of degree three.

Another extension of Formula 2.3.1 is to increase the order of multistep($p = k - 1$). Let us consider $p = 2$, i.e. $k = 3$, then the interpolating polynomial is now moved two tabular points at a time to the right instead of one as for $p = 1$. The simplest case possible for odd q is $q = 5$ corresponding to Formula 3.5.1 or the seven-point double-step formula of the first kind. Here the Hermitian polynomial passes through three points instead of two as in the single-step formula. For symmetry the seven points used may be ordered as $\{x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3\}$. The formula thus obtained is exact to a polynomial of degree four with a coefficient matrix F in (3.53) being

$$(4.4) \quad \frac{1}{48} \begin{bmatrix} 0 & 0 & 0 & 48 & 0 & 0 & 0 \\ -1 & 8 & -37 & 0 & 37 & -8 & 1 \\ 1 & -8 & 47 & -80 & 47 & -8 & 1 \\ 1 & -8 & 13 & 0 & -13 & 8 & -1 \\ -1 & 8 & -23 & 32 & -23 & 8 & -1 \end{bmatrix}$$

For all double-step formulae the total number of points will be odd. Similarly as for single-step formula we may increase the value of q to obtain formulae which will exact to a polynomial of degree four but which use more points in evaluating the derivative. Likewise it is possible to extend the double-step formulae such that the interpolating polynomial moves three tabular points at a time to the right, and these formulae may be called the triple-step formulae of the first kind. With $q = 5$ we have the eight-point triple-step formula of the first kind, or Formula 4.5.1, with the coefficient matrix

$$(4.5) \quad \frac{1}{6912} \begin{bmatrix} -27 & 216 & -918 & 4185 & 4185 & -918 & 216 & -27 \\ 18 & -144 & 756 & -8586 & 8586 & -756 & 144 & -18 \\ 120 & -960 & 3888 & -3048 & -3048 & 3888 & -960 & 120 \\ -80 & 640 & -3232 & 7056 & -7056 & 3232 & -640 & 80 \\ -48 & 384 & -864 & 528 & 528 & -864 & 384 & -48 \\ 32 & -256 & 832 & -1440 & 1440 & -832 & 256 & -32 \end{bmatrix}$$

The eight points may be ordered $\{x_i, i = -3.5 \text{ to } 3.5\}$ where subset $\{x_i, i = -3.5 \text{ to } .5\}$ is used in evaluating the derivative at $x_{-1.5}$ while the subset $\{x_i, i = -.5 \text{ to } 3.5\}$ is used in evaluating the derivative at $x_{1.5}$. The Hermitian polynomial passes through $x_i, i = -1.5, -.5, .5, 1.5$. This formula is exact to a polynomial of degree four.

Two examples of quadruple-step formulae of the first kind, Formulae 5.3.1 and 5.5.1 may be found in Table II.1 of Appendix II.

4.2 Continuous First and Second Derivatives

Formulae with continuity of the first two derivatives may be called the formulae of the second kind. With $k = 2$, the simplest case possible with odd q is Formula 2.5.2 which was expressed in non-symmetric form by Snyder [34] and subsequently in symmetric form by Charmonman and Wojtiw [10]. This formula may be called the six-point single-step formula of the second kind with coefficient matrix

$$(4.6) \quad \frac{1}{768} \begin{bmatrix} 9 & -75 & 450 & 450 & -75 & 9 \\ -26 & 162 & -1124 & 1124 & -162 & 26 \\ -40 & 312 & -272 & -272 & 312 & -40 \\ 144 & -848 & 1824 & -1824 & 848 & -144 \\ 16 & -48 & 32 & 32 & -48 & 16 \\ -160 & 800 & -1600 & 1600 & -800 & 160 \end{bmatrix}$$

This formula is exact to a polynomial of degree four.

Like the formulae of the first kind, the formulae of the second kind may be extended by increasing values of q and k . An example of such an extension is the eight-point triple-step formulae of the second kind or Formula 4.5.2, which has the coefficient matrix

$$(4.7) \quad \frac{1}{20736} \begin{bmatrix} -162 & 1053 & -3483 & 12960 & 12960 & -3483 & 1053 & -162 \\ 135 & -918 & 3564 & -27783 & 27783 & -3564 & 918 & -135 \\ 756 & -4860 & 15228 & -11124 & -11124 & 15228 & -4860 & 756 \\ -636 & 4296 & -16032 & 31068 & -31068 & 16032 & -4296 & 636 \\ -448 & 2672 & -5328 & 3104 & 3104 & -5328 & 2672 & -448 \\ 400 & -2592 & 7360 & -11920 & 11920 & -7360 & 2592 & -400 \\ 64 & -320 & 576 & -320 & -320 & 576 & -320 & 64 \\ -64 & 384 & -1024 & 1600 & -1600 & 1024 & -384 & 64 \end{bmatrix}$$

The interpolating polynomial here moves three tabular points at a time, while the Hermitian polynomial passes through x_i , $i = -1.5, -.5, .5, 1.5$. This formula is exact to a polynomial of degree four.

4.3 Continuous First Three Derivatives

Formulae with continuity up to the third derivative may be called formulae of the third kind. Three examples of such formulae are given in Table II.3 of Appendix II, one single-step formula and two

double-step formulae. Formula 2.7.3, is single-step and exact to a polynomial of degree six, while Formulae 3.7.3 and 3.9.3, are double-step and exact to a polynomial of degree six and eight respectively.

4.4 Generation of Desk-Calculator Tables

Tables of continuous piecewise interpolation coefficients for desk-calculator use may be obtained on an electronic computer. Consider the set

$R = \{x_j, j = -\frac{n}{2}, -\frac{n}{2} + 1, \dots, \frac{n}{2}, \text{ such that } x_j = 1 \text{ if } j = -\frac{n}{2} + i \text{ and } x_j = 0 \text{ otherwise}\}$ where i is one of the values $0, 1, \dots, n$. For the coefficient matrix F in (3.53) the individual and unit contribution of $y_{-\frac{n}{2} + i}$ for an appropriate x is obtained by considering the set R and solving equation (3.53) for all values of e_j i.e.

$$(4.8) \quad c_i(x) = \sum_{j=0}^{2m+k-1} e_j x^j .$$

The quantity $c_i(x)$ is called the interpolating coefficient [34]. Since there are $(n + 1)$ of these unit contributions we may interpolate in any set $\{x_i, i = -\frac{n}{2}, \dots, \frac{n}{2}\}$ by

$$(4.9) \quad H_n(x) = \sum_{i=0}^n c_i(x) y_{-\frac{n}{2} + i} .$$

Desk-calculator tables can be generated in any increment of x from $x = -.5$ to $x = .5$ for single-step formulae; from $x = -1.0$ to $x = 1.0$ for double-step formulae; from $x = -1.5$ to $x = 1.5$ for triple-step formulae and from $x = -2.0$ to $x = 2.0$ for the quadruple-step formulae. An example of coefficients, for Formula 2.5.2, as generated on a computer is given in Table III.3 of Appendix III.

CHAPTER V

TEST RESULTS AND CONCLUSIONS

A practical test of the method of continuous piecewise interpolation formulae is difficult to design. In practical applications the true function is not known, since if it were there would be little point in having an interpolation formula because any desired values could be computed from the true function. Thus a test to decide which continuous piecewise interpolation formula would give the best approximation to intermediate values for measurements of physical quantities would be impossible to construct. Therefore the various continuous piecewise interpolation formulae are tested on known functions.

5.1 Test Functions and Procedure

The known functions considered were

$$(5.1) \quad y = x^i \quad \text{for } i = 6, 7, 10$$

and

$$(5.2) \quad y = e^x$$

For (5.1) interpolation was carried out in the interval $[1.025, 2]$ in increments of .025, while for (5.2) in the intervals $[10.025, 11]$ and $[25.025, 26]$ also in increments of .025.

The number of multiplications required in the application of the formula can be minimized by using the mirror-image property of the coefficient matrix F in (3.53). In other words, with $F = \{f_{ij}\}$ denoting the coefficient matrix,

$$(5.3) \quad e_i = \sum_{j=1}^{n+1} f_{ij} y_{-\frac{n}{2} + j-1} \quad i = 1, 2, \dots, (2m+k)$$

and symmetry in F , (5.3) may be written

$$(5.4) \quad e_i = \begin{cases} \sum_{j=1}^{t-1} f_{ij} (y_{-\frac{n}{2} + j-1} \pm y_{\frac{n}{2} + 1-j}) & \text{for } (n+1) \text{ even} \\ f_{it} y_0 + \sum_{j=1}^{t-1} f_{ij} (y_{-\frac{n}{2} + j-1} \pm y_{\frac{n}{2} + 1-j}) & \text{for } (n+1) \text{ odd} \end{cases}$$

where $t = \left\lfloor \frac{n+1}{2} \right\rfloor + 1$ with plus and minus signs depending upon the coefficient matrix F . The term $f_{it} y_0$ in (5.4) is zero for i even and $(n+1)$ odd. Also the expression

$$\sum_{j=1}^{t-1} f_{ij} (y_{-\frac{n}{2} + j-1} \pm y_{\frac{n}{2} + 1-j})$$

for $(n+1)$ odd in (5.4) is zero for $i = 1$. Thus for $(n+1)$ even (5.4) requires $(n+1) \div 2$ multiplications and n additions (algebraic) as opposed to $(n+1)$ multiplications and n additions in (5.3). For $(n+1)$ odd, in the worst case (5.4) requires $(\frac{n+1}{2}) + 1$ multiplications and n additions as opposed to $(n+1)$ multiplications and n additions in (5.3). Therefore

computation of the e_i 's by use of (5.4) can be made to be about twice as efficient as in using (5.3). Furthermore a much more efficient way to evaluate any polynomial is to factor out the x 's. For example, for

$$(5.5) \quad Y(x) = \sum_{i=1}^6 a_i x^{i-1}$$

we may write

$$(5.6) \quad Y(x) = a_1 + x(a_2 + x(a_3 + x(a_4 + x(a_5 + xa_6))))).$$

(5.6) requires only five multiplications and five additions as opposed to $\sum_{i=1}^5 i = 15$ multiplications and five additions for (5.5). In some machines where one multiplication takes much longer than one addition then addition time can be neglected and the computation of $Y(x)$ in (5.6) requires about 1/3 of that for (5.5). In some other machines one multiplication takes about as long as one addition and the computation of $Y(x)$ in (5.6) costs about half that in computing $Y(x)$ in (5.5). The number of multiplications required for computing each formula by the above procedures are given in Table IV.1 of Appendix IV.

The test procedure was to apply the various formulae to the given function and obtain interpolated values for this function in increments of .025. Since we know the true solution, we may then obtain the relative error for each interpolated value. We now calculate the largest relative error for each formula and use this as a measure of how good an approximation has been obtained. The results of (5.1) are given in Table IV.2 of Appendix IV, while those of (5.2) in Table IV.3 of the

same appendix.

5.2 Conclusions

The following observations may be drawn from the results of this study:

1. Comparison of the exact result to the floating-point machine derivation revealed that the actual errors in the floating-point result were so small that it is possible to convert the floating-point result into exact integer form.
2. Many new formulae presented in this thesis gave a better approximation than Snyder's Formula 2.3.1 [32], but these required a larger number of multiplications. For example, Formula 2.5.1 gave a more accurate result for the test examples than Formula 2.3.1 in both Tables IV.2 and IV.3 of Appendix IV. However Formula 2.5.1 requires 15 multiplication as opposed to 11 for Formula 2.3.1. Similarly for Snyder's Formula 2.5.2 [34], there are cases where a more accurate approximation is achieved requiring a larger number of multiplications i.e. Formula 2.7.2 in Tables IV.2 and IV.3. Formula 3.7.1 requires the same number of multiplications and yet it gives a more accurate result than Formula 2.5.2 for the test examples. No formula was found requiring a lower number of points and giving a better approximation than Formula 2.5.2.
3. Increasing the number of points in evaluating the derivative i.e. q , produces a better approximation up to a certain value of q , after which no improvement results in the approximation. This is obvious if we consider Formulae 2.3.1, 2.5.1, 2.7.1, 2.9.1, and 2.11.1 for x^6 and x^7 in Table IV.2, and e^x in Table IV.3. The same degree of accuracy may

be achieved using either Formula 2.5.1 or Formulae 2.7.1, 2.9.1, or 2.11.1. Thus when more than one formula give the same accuracy we should use the formula requiring the smallest number of multiplications.

4. Formulae of the second kind gave a lower maximum relative error than those of the first kind, as is obvious for Formulae 2.7.1 and 2.7.2 in Tables IV.2 and IV.3. Comparing formulae of the second kind with those of the third kind, no noticeable improvement resulted in the maximum relative error, but an increase in the number of multiplications was observed. This would imply that formulae of the second kind can be used instead of the first kind to obtain a better approximation, and instead of the third kind to obtain a decrease in the number of multiplications.

5. An increase in the order of the multistep for constant values of q and m does not necessarily produce an improvement in the maximum relative error. An improvement may result depending upon whether p is even or odd, the value of q , and the value of i in x^i . If we consider the maximum relative errors for Formulae 2.3.1, 3.3.1, 4.3.1, and 5.3.1, for an increase in p from one to three no improvement results, while for an increase in p from 3 to 4 an improvement is observed for Table IV.2 only. If on the other hand we consider Formulae 2.5.1 and 3.5.1 an improvement results in the maximum relative error with an increase in the order of the multistep.

6. We may obtain the formulae requiring the lowest number of multiplications for the lowest maximum relative error for Tables IV.2 and IV.3 of Appendix IV. These for Table IV.2 for x^6 , x^7 , and x^{10} are Formulae 3.7.2, 3.9.3, and 3.9.2 respectively, while for Table IV.3 in the intervals $[10.025, 11]$ and $[25.025, 26]$ the formulae are Formulae 3.9.3 and 2.7.2.

5.3 Suggestions for Future Research

1. A computer program, say in Fortran IV, may be written such that all operations are performed in integer notation. This would mean that all operations would have to be carried out under the lowest common denominator. This would result in the coefficient matrix being in integer form. Special care must be given to the problem of overflow in many parts of the program.

2. The various continuous piecewise interpolation formulae may be tested on measurements of physical quantities such as streamflow, precipitation, etc. For any particular engineering problem we may make the measurements at some intermediate point in addition to the equidistance set of points. Then test the various formulae available to find the formula which requires the smallest number of multiplication from among those giving results with an acceptable degree of accuracy.

BIBLIOGRAPHY

1. Ahlberg, J.H. and Nilson, E.N., "Convergence Properties of the Spline Fit", Notices Am. Math. Soc., abs. 1961, 61T-219, Vol. 8, No. 5, October 1961, p. 440.
2. Ahlberg, J.H. and Nilson, E.N., "Convergence Properties of the Spline Fit", Intern. Congr. Math., Stockholm, 1962.
3. Ahlberg, J.H. and Nilson, E.N., "Convergence Properties of the Spline Fit", J. Soc. Indust. Appl. Math., 11, 1963, pp. 95-104.
4. Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., "Fundamental Properties of Generalized Splines", Proc. Natl. Acad. Sci. U.S., 52, 1964, pp. 1412-19.
5. Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., "Extremal, Orthogonality, and Convergence Properties of Multi-Dimensional Splines", Notices Am. Math. Soc., June, 1964, 64T-339, Vol. 11, No. 4, p. 468.
6. Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., "Best Approximation and Convergence Properties of Higher Order Spline Approximations", J. Math. Mech., 14, 1965, pp. 231-244.

7. Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., "Extremal, Orthogonality, and Convergence Properties of Multi-Dimensional Splines", J. Math. Anal. Appl., 11, 1965, pp. 27-48.
8. Ahlberg, J.H., Nilson, E.N., and Walsh, J.L., The Theory of Splines and Their Applications, Academic Press, New York, 1967.
9. Berezin, I.S., and Zhidkov, N.P., Computing Methods, Pergamon Press, Addison-Wesley Publishing Co. Inc., Vol. 1, 1965.
10. Charmonman, S., and Wojtiw, L., Discussion of "Extended Continuous Interpolation", by W.M. Snyder, Journal of the Hydraulics Division, ASCE, Vol. 94, No. HY3, Proc. Paper 5920, May, 1968, pp. 810-13.
11. Davis, P.J., Interpolation and Approximation, Blaisdell Publishing Co., 1965.
12. De Boor, "Bicubic Spline Interpolation", Notices Am. Math. Soc., 579-24, Vol. 8, No. 2, April 1961, p. 162.
13. De Boor, "Bicubic Spline Interpolation", J. Math Phys., 41, 1962, pp. 212-218.

14. Falkoff, A.D., and Iverson, K.E., "APL/360 Terminal System",
ACM Symposium on Interactive Systems for Experimental
Applied Mathematics, Aug., 1967.
15. Forsythe, G.E., and Moler, C.B., Computer Solution of Linear
Algebraic Systems, Prentice-Hall, Inc., Englewood Cliffs,
N.J., 1967.
16. Froberg, C.E., Introduction to Numerical Analysis, Addison-
Wesley Publishing Co. Inc., 1965.
17. Hilderbrand, F.B., Introduction to Numerical Analysis, McGraw-
Hill Book Co., New York, 1956.
18. Holladay, J.C., "Smoothest Curve Approximation", Math. Tables
Aids Computation, 11, pp. 233-43.
19. Isaacson, E., and Keller, H.B., Analysis of Numerical Methods,
John Wiley & Sons, Inc., New York, 1966.
20. Iverson, K.E., A Programming Language, John Wiley & Sons, Inc.,
New York, 1962.
21. Kopal, Z., Numerical Analysis, Chapman and Hall Ltd., London,
1955.
22. Kunz, K.S., Numerical Analysis, McGraw-Hill Book Co. Ltd.,
New York, 1957.

23. Kurtz, M., "Discussion of Continuous Parabolic Interpolation",
by W.M. Snyder, Journal of the Hydraulics Division,
ASCE, Vol. 88, No. HY1, Proc. Paper 3044, Jan., 1962,
pp. 139-143.
24. Ralston, A., A First Course in Numerical Analysis, McGraw-Hill
Book Co., New York, 1965.
25. Scarborough, J.B., Numerical Mathematical Analysis, The John
Hopkins Press, Baltimore, Sixth Edition, 1966.
26. Schoenberg, I.J., "Contributions to the Problem of Approximation
of Equidistant Data by Analytic Functions", Quart.
Appl. Math., 4, 1946, pp. 45-99, 112-141.
27. Schoenberg, I.J., and Whitney, A., "Sur la positivité des
déterminants de translations de fonctions de fréquence
de Pólya avec une application au problème d'inter-
polation par les fonctions "spline", Compt. Rend.,
1949, pp. 1996-8.
28. Schoenberg, I.J., and Whitney, A., "On Polya frequency functions,
III, Trans. Am. Math. Soc., 74, 1953, pp. 246-259.
29. Sharma and Meir, "Convergence of Spline Functions", Notices
Am. Math. Soc., November, 1964, 647-496, Vol. 11,
No. 7, p. 768.

30. Sharma and Meir, "Degree of Approximation of Spline Interpolation", J. Math. Mech., 15, pp. 759-67.
31. Singer, J., Elements of Numerical Analysis, Academic Press, New York, 1964.
32. Snyder, W.M., "Continuous Parabolic Interpolation", Journal of the Hydraulics Division, ASCE, Vol. 87, No. HY4, Proc. Paper 2865, July, 1961, pp. 99-111.
33. Snyder, W.M., Closure of "Continuous Parabolic Interpolation" by W.M. Snyder", Journal of the Hydraulics Division, ASCE, Vol. 88, No. HY4, Proc. Paper 3209, July, 1962, pp. 265-274.
34. Snyder, W.M., "Extended Continuous Interpolation", Journal of the Hydraulics Division, ASCE, Vol. 93, No. HY5, Proc. Paper 5467, Sept., 1967, pp. 261-280.
35. Wilkinson, J.H., "Error Analysis of Direct Methods of Matrix Inversion", J. Assoc. Comp. Mach., 8, 1961, pp. 281-330.
36. Wilkinson, J.H., Rounding Errors in Algebraic Processes, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

APPENDIX I

APL\360 FUNCTIONS

The following pages contain a description and a listing of APL\360 functions used throughout this thesis. Two functions SIGDIG and EFT are omitted since they are found in LIBRARY 1. The function SIGDIG controls the number of significant digits, between one and seventeen, in the final result, while the function EFT enables us to output the results with a different number of significant digits in each column. For example, if we have four columns we may output the first column say with two significant digits while the other three with eight significant digits as in Table III.3 of Appendix III.

CPI B corresponds to Algorithm 3.1 of subsection 3.4 in Chapter III where B is the vector of the three parameters in (4.1). This function uses INVERSE M to calculate the inverse of a non-singular square matrix M by Gaussian elimination.

A VAL Δ BETA B and A E Δ NORM Δ CPIF B give the values of beta in (3.83) and $\|\alpha F - G\|_E$ respectively, A is the vector of the three parameters in (4.1) while B is the reciprocal of the coefficient of the formula. This coefficient is given for each formula in Appendix II.

The function C DESK Δ COEFF B is used in generating the desk-coefficients of Formula 2.5.2 given in Table III.3 of Appendix III. C is a vector of three components representing the starting point, the number of rows, and the group of three columns desired in the table of coefficients. B is the vector of the three parameters in (4.1). This function calls Z POLY T to calculate the value of a polynomial with its coefficients represented by

the vector T at the point Z .

The function `BAPPLICΔCPIF A` is used in obtaining the values of Tables IV.2 and IV.3 (i.e. the results of (5.1) and (5.2) respectively). B is a vector of three components representing the first point of interpolation, the increment, and the i^{th} power of x for Table IV.2. For the results of Table IV.3 the third component of B is replaced by $\odot x$ (for $\exp x$) in the function and deleted from B in the header. A is the vector of the three parameters in (4.1). The largest relative error for each formula is obtained in the output.

∇APPLICΔCPIF[□]∇

```

∇ Q←B APPLICΔCPIF A;C;D;E;F;G;H;I;J;K;L;M;N;O;P;R;S;T;U
  ;V;W;X;Y;Z
[1] J←⌊((ρA)÷3)
[2] C←15ρ0
[3] Q←(J,1)ρP←K←M←N←0
[4] U←CPI V←A[P+13]
[5] X←(B[1]+B[2]×110×(4-(V[1]=4)))*B[3]
[6] W←(-(0.5+((V[1]-2)×((R←0.25)+0.25))))+
  0.1×110×(V[1]-1)
[7] Z←(B[1]-((⌊(V[2]÷2))+1)×B[2]×10)+(B[2]×10×1((V[1]-1)+
  V[2]))
[8] D←Z*B[3]
[9] E←2⌊((ρU)[2])
[10] F←⌊((ρU)[1])÷2
[11] G←⌊((ρU)[2])÷2
[12] →(E=0)/15
[13] C[H]←U[H←((2×1(F+1))-1);1(G+1)]+.×(((D[1G]+D[1+φ(1G)+
  G])),D[G+1])
[14] →17,C[H]←U[H←2×1F;1G]+.×(D[1G]-D[1+φ(1G)+G])
[15] C[H]←U[H←((2×1F)-1);1G]+.×(D[1G]+D[φ(1G)+G])
[16] C[H]←U[H←2×1F;1G]+.×(D[1G]-D[φ(1G)+G])
[17] O←W POLY C
[18] →((Y←⌊4÷(V[1]-1))=4)/20
[19] →21,I←1
[20] I←1+R←0
[21] →(Y=I)/30
[22] L←(Z←0.25+R+Z)*B[3]
[23] →(E=0)/26
[24] C[H]←U[H←((2×1(F+1))-1);1(G+1)]+.×(((L[1G]+L[1+φ(1G)+
  G])),L[G+1])
[25] →28,C[H]←U[H←2×1F;1G]+.×(L[1G]-L[1+φ(1G)+G])
[26] C[H]←U[H←((2×1F)-1);1G]+.×(L[1G]+L[φ(1G)+G])
[27] C[H]←U[H←2×1F;1G]+.×(L[1G]-L[φ(1G)+G])
[28] O←O,W POLY C
[29] →21,I←I+1
[30] Q[M←M+1;1]←(S[T])÷X[T←R1↑/R←↓S←(X-O)]
[31] P←P+3
[32] →(J>K←K+1)/4

```

∇

∇POLY[□]∇

```

∇ Q←Z POLY T
[1] Q←+/(((ρZ),ρT)ρT)×Z°. *-1+1ρT

```

∇


```

      ∇ CPI[ ] ∇
    ∇ Q ← CPI B; K; M; N; D; C; A; V
  [1] → ((2 × B[3]) + B[1]) ≤ ((B[1] + B[2]) - 1) / 3
  [2] → 0, ρ Q ← ('NO UNIQUE SOLUTION EXISTS')
  [3] A ← (K, K ← (N ← B[1]) + 2 × D ← B[3]) ρ 0
  [4] A[D + 1; N; ] ← ((1; N) - 0.5 × N + 1) × V ← 1 + 1; K
  [5] A[1; D; ] ← φ[1] (1 × (-1; D) × V) × A[D + N + 1; D; ] ← ((Q(K, D) ρ 1; D) × (1; D) × V) × (0.5 × N - 1) × (-1; D) × V
  [6] C ← (1; K) × (1; N + M - 1) + D + 0.5 × 1 - M ← B[2]
  [7] C[1; D; 1; M] ← φ[1] C[D + N + 1; D; N - 1 - 1; M] ← (Q(M, D) ρ 1; D) × (INVERSE((1; M) - 0.5 × M + 1) × 1 + 1; M)[1 + 1; D; ]
  [8] Q ← (INVERSE A) × C
    ∇

```

```

      ∇ INVERSE[ ] ∇
    ∇ R ← INVERSE M; N; A; K; I; V
  [1] Y ← (, M[; A + N] ← A × A, (, M ← M[; A, A]), (, A ← 1; N ← (ρ M)[K + 1])
  [2] → ((1; E - 30) < |M[K; K] + 0 × ρ((, M[K, I; ] ← M[I, K; ]), (, I ← (K - 1) + V 1; [ / V ← |M[(K - 1) + 1; N + 1 - K; K])|) / 4
  [3] → 0, ρ R ← ('DETERMINANT IS ZERO')
  [4] Y ← 0 × ((, M ← M - (M[; K] × K ≠ A) × M[K; ]), (, M[K; ] ← M[K; ] ÷ M[K; K])
  [5] → (N ≥ K + K + 1) / 2
  [6] R ← M[; A + N]
    ∇

```

```

      ∇ VAL Δ BETA[ ] ∇
    ∇ Q ← A VAL Δ BETA B; U; V
  [1] Q ← U[1; [ / V ← |U ← ((1; 0.5 + (CPI A) × B) - ((CPI A) × B))]
    ∇

```

```

      ∇ E Δ NORM Δ CPI F[ ] ∇
    ∇ Q ← A E Δ NORM Δ CPI F B; C
  [1] C ← ((1; 0.5 + (CPI A) × B) - ((CPI A) × B))
  [2] Q ← (+ / (C × C)) × 0.5
    ∇

```



```

      VDESKΔCOEFF[□]V
V Q←C DESKΔCOEFF B;A;Z;C0;C1;C2
[1] A←CPI B
[2] Z←(-C[1])+0.01×C[2]
[3] C0←Z POLY A[;C[3]+1]
[4] C1←Z POLY A[;C[3]+2]
[5] C2←Z POLY A[;C[3]+3]
[6] ('      X      C';C[3]+1;'      C';C[
      3]+2;'      C';C[3]+3)
[7] Q←(10 2 ,6ρ 16 8)EFT(4,C[2])ρZ,C0,C1,C2
V

```


APPENDIX II

COEFFICIENT MATRICES OF THE FORMULAE

The various continuous piecewise interpolation formulae are classified according to the highest order of the derivative required for continuity and are called formulae of the m^{th} kind. The coefficient matrix F of (3.53) for each formula is given in integer form in this appendix. The formulae of the first kind are given in Table II.1, the formulae of the second kind in Table II.2, and the formulae of the third kind in Table II.3.

TABLE II.1
FORMULAE OF THE 1st KIND

FORMULA 2.3.1

COEFFICIENT OF MATRIX $1 \div 16$

$$\begin{array}{cccc} \bar{1} & 9 & 9 & \bar{1} \\ 2 & \bar{22} & 22 & \bar{2} \\ 4 & \bar{4} & \bar{4} & 4 \\ \bar{8} & 24 & \bar{24} & 8 \end{array}$$

FORMULA 2.5.1

COEFFICIENT OF MATRIX $1 \div 96$

$$\begin{array}{cccccc} 1 & \bar{9} & 56 & 56 & \bar{9} & 1 \\ \bar{2} & 14 & \bar{128} & 128 & \bar{14} & 2 \\ \bar{4} & 36 & \bar{32} & \bar{32} & 36 & \bar{4} \\ 8 & \bar{56} & 128 & \bar{128} & 56 & \bar{8} \end{array}$$

FORMULA 2.7.1

COEFFICIENT OF MATRIX $1 \div 480$

$$\begin{array}{ccccccccc} \bar{1} & 10 & \bar{54} & 285 & 285 & \bar{54} & 10 & \bar{1} \\ 2 & \bar{16} & 72 & \bar{630} & 630 & \bar{72} & 16 & \bar{2} \\ 4 & \bar{40} & 216 & \bar{180} & \bar{180} & 216 & \bar{40} & 4 \\ \bar{8} & 64 & \bar{288} & 600 & \bar{600} & 288 & \bar{64} & 8 \end{array}$$

TABLE II.1 (continued)

FORMULA 2.9.1

COEFFICIENT OF MATRIX 1÷6720

3	-35	200	-840	4032	4032	-840	200	-35	3
-6	58	-272	1008	-8736	8736	-1008	272	-58	6
-12	140	-800	3360	-2688	-2688	3360	-800	140	-12
24	-232	1088	-4032	8064	-8064	4032	-1088	232	-24

FORMULA 2.11.1

COEFFICIENT OF MATRIX 1÷20160

-2	27	-175	750	-2700	12180	12180	-2700	750	-175	27
4	-46	250	-900	3000	-26040	26040	-3000	900	-250	46
8	-108	700	-3000	10800	-8400	-8400	10800	-3000	700	-108
-16	184	-1000	3600	-12000	23520	-23520	12000	-3600	1000	-184

TABLE II.1 (continued)

FORMULA 3.3.1

COEFFICIENT OF MATRIX $1 \div 8$

0	0	8	0	0
1	-6	0	6	-1
-1	8	-14	8	-1
-1	2	0	-2	1
1	-4	6	-4	1

FORMULA 3.5.1

COEFFICIENT OF MATRIX $1 \div 48$

0	0	0	48	0	0	0
-1	8	-37	0	37	-8	1
1	-8	47	-80	47	-8	1
1	-8	13	0	-13	8	-1
-1	8	-23	32	-23	8	-1

FORMULA 3.7.1

COEFFICIENT OF MATRIX $1 \div 240$

0	0	0	0	240	0	0	0	0
1	-9	46	-189	0	189	-46	9	-1
-1	9	-44	231	-390	231	-44	9	-1
-1	9	-46	69	0	-69	46	-9	1
1	-9	44	-111	150	-111	44	-9	1

TABLE II.1 (continued)

FORMULA 4.3.1

COEFFICIENT OF MATRIX 1÷3456

81	-459	2106	2106	-459	81
-54	378	-4320	4320	-378	54
-360	1944	-1584	-1584	1944	-360
240	-1616	3648	-3648	1616	-240
144	-432	288	288	-432	144
-96	416	-768	768	-416	96

FORMULA 4.5.1

COEFFICIENT OF MATRIX 1÷6912

-27	216	-918	4185	4185	-918	216	-27
18	-144	756	-8586	8586	-756	144	-18
120	-960	3888	-3048	-3048	3888	-960	120
-80	640	-3232	7056	-7056	3232	-640	80
-48	384	-864	528	528	-864	384	-48
32	-256	832	-1440	1440	-832	256	-32

FORMULA 5.3.1

COEFFICIENT OF MATRIX 1÷576

0	0	0	576	0	0	0
-24	136	-488	0	488	-136	24
12	-80	500	-864	500	-80	12
30	-158	226	0	-226	158	-30
-15	94	-241	324	-241	94	-15
-6	22	-26	0	26	-22	6
3	-14	29	-36	29	-14	3

TABLE II.1 (continued)

FORMULA 5.5.1

COEFFICIENT OF MATRIX 1÷1152

0	0	0	0	1152	0	0	0	0
8	-64	272	-960	0	960	-272	64	-8
-4	32	-160	992	-1720	992	-160	32	-4
-10	80	-316	432	0	-432	316	-80	10
5	-40	188	-472	638	-472	188	-40	5
2	-16	44	-48	0	48	-44	16	-2
-1	8	-28	56	-70	56	-28	8	-1

TABLE II.2
FORMULAE OF THE 2nd KIND

FORMULA 2.5.2

COEFFICIENT OF MATRIX 1÷768

9	-75	450	450	-75	9
-26	162	-1124	1124	-162	26
-40	312	-272	-272	312	-40
144	-848	1824	-1824	848	-144
16	-48	32	32	-48	16
-160	800	-1600	1600	-800	160

FORMULA 2.7.2

COEFFICIENT OF MATRIX 1÷11520

-28	275	-1377	6890	6890	-1377	275	-28
80	-614	2430	-16300	16300	-2430	614	-80
128	-1240	5832	-4720	-4720	5832	-1240	128
-448	3376	-12528	23840	-23840	12528	-3376	448
-64	560	-1296	800	800	-1296	560	-64
512	-3680	11232	-18880	18880	-11232	3680	-512

FORMULA 3.5.2

COEFFICIENT OF MATRIX 1÷96

0	0	0	96	0	0	0
-3	20	-79	0	79	-20	3
4	-28	124	-200	124	-28	4
4	-24	36	0	-36	24	-4
-6	40	-106	144	-106	40	-6
-1	4	-5	0	5	-4	1
2	-12	30	-40	30	-12	2

TABLE II.2 (continued)

FORMULA		3,11,2											
		COEFFICIENT OF MATRIX											
		1÷403200											
0	0	0	0	0	0	0	0	0	0	0	0	0	0
132	0	-1625	9648	-37875	116500	-352234	0	352234	-116500	37875	0	0	-9648
-172	1625	-132	-12312	45625	-132500	471034	-747600	471034	-132500	45625	-12312		
-184	2125	-172	-13216	50750	-143000	176468	0	-176468	143000	-50750	13216		
264	-2250	184	18704	-68250	187000	-361268	453600	-361268	187000	-68250	18704		
52	-3250	264	3568	-12875	26500	-25834	0	25834	-26500	12875	-3568		
-92	625	-52	-6392	22625	-54500	91834	-109200	91834	-54500	22625	-6392		
	1125	-92											
	1125												

FORMULA		4,5,2											
		COEFFICIENT OF MATRIX											
		1÷20736											
-162	1053	-3483	12960	12960	-3483	1053	-162						
135	-918	3564	-27783	27783	-3564	918	-135						
756	-4860	15228	-11124	-11124	15228	-4860	756						
-636	4296	-16032	31068	-31068	16032	-4296	636						
-448	2672	-5328	3104	3104	-5328	2672	-448						
400	-2592	7360	-11920	11920	-7360	2592	-400						
64	-320	576	-320	-320	576	-320	64						
-64	384	-1024	1600	-1600	1024	-384	64						

TABLE II.2 (continued)

FORMULA	4,7,2	COEFFICIENT OF MATRIX									
		1÷2488320		1546209		142155		413343		1546209	
4131	-34992	142155	-413343	1546209	1546209	142155	-413343	1546209	-413343	142155	-34992
-3402	29160	-123930	417150	-3293622	3293622	-123930	417150	-3293622	-417150	123930	-29160
-19332	153296	-656100	1803924	-1291788	-1291788	-656100	1803924	-1291788	1803924	-656100	163296
16056	-137376	579960	-1872936	3536136	-3536136	579960	-1872936	3536136	1872936	-579960	137376
11664	-96768	360720	-618192	342576	342576	360720	-618192	342576	-618192	360720	-96768
-10208	86400	-349920	846240	-1302048	1302048	-349920	846240	-1302048	-846240	349920	-86400
-1728	13824	-43200	63936	-32832	-32832	-43200	63936	-32832	63936	-43200	13824
1664	-13824	51840	-115584	169344	-169344	51840	-115584	169344	115584	-51840	13824

FORMULA	5,5,2	COEFFICIENT OF MATRIX									
		1÷55296		52992		0		-20288		6400	
0	-6400	0	55296	0	52992	0	55296	0	0	0	0
992	20288	-52992	0	-20288	6400	-52992	0	-20288	-992	6400	-992
-592	-14272	58496	-95200	-14272	3968	58496	-95200	-14272	-592	3968	-592
-1392	-26016	31104	0	26016	-8832	-31104	0	26016	1392	-8832	1392
840	-5568	18912	49584	18912	-5568	18912	49584	18912	840	-5568	840
438	-2640	6180	0	-6180	2640	6180	0	-6180	-438	2640	-438
-273	1752	-5052	-10470	-5052	1752	-5052	-10470	-5052	-273	1752	-273
-38	208	-452	0	452	-208	-452	0	452	38	-208	38
25	-152	412	790	-412	-152	412	790	-412	25	-152	25

TABLE II.3
FORMULAE OF THE 3rd KIND

FORMULA 2.7.3

COEFFICIENT OF MATRIX 1÷92160

-225	2205	-11025	55125	55125	-11025	2205	-225
818	-6158	23178	-136630	136630	-23178	6158	-818
1036	-9980	46764	-37820	-37820	46764	-9980	1036
-5720	41960	-145080	265480	-265480	145080	-41960	5720
-560	4720	-10800	6640	6640	-10800	4720	-560
12640	-89248	269280	-450080	450080	-269280	89248	-12640
64	-320	576	-320	-320	576	-320	64
-11392	79744	-239232	398720	-398720	239232	-79744	11392

FORMULA 3.7.3

COEFFICIENT OF MATRIX 1÷11520

0	0	0	0	11520	0	0	0	0
83	-690	2890	-9802	0	9802	-2890	690	-83
-139	1176	-4756	16424	-25410	16424	-4756	1176	-139
-125	990	-3670	4870	0	-4870	3670	-990	125
269	-2232	8492	-18184	23310	-18184	8492	-2232	269
49	-342	878	-926	0	926	-878	342	-49
-169	1368	-4828	9704	-12150	9704	-4828	1368	-169
-7	42	-98	98	0	-98	98	-42	7
39	-312	1092	-2184	2730	-2184	1092	-312	39

APPENDIX III

VALUES AND DESK-COEFFICIENTS

In this appendix three tables are included. In the first table Table III.1 the various continuous piecewise interpolation formulae are summarized. In the second table Table III.2 the values of $\beta = \min \epsilon_{ij}$ (3.83) and $\|\alpha F - G\|_E$ are given for each formula. The value of β is obtained using the function VAL Δ BETA while the value of $\|\alpha F - G\|_E$ is calculated by E Δ NORM Δ CPIF. The third table Table III.3 is the desk-coefficients for Formula 2.5.2 in increments of .01. These coefficients are generated by using the functions DESK Δ COEFF, POLY, EFT, CPI and INVERSE. There are six coefficients, C_i , $i = 1, 2, \dots, 6$, for each value of $x = -.5$ to $x = .5$.

TABLE III.1

SUMMARY OF THE FORMULAE

<u>FORMULA</u>	<u>TOTAL NUMBER OF POINTS USED</u>	<u>HIGHEST ORDER OF CONTINUOUS DERIVATIVE</u>	<u>ORDER OF THE MULTISTEP</u>	<u>LOWEST DEGREE OF POLYNOMIAL USED</u>
2.3.1	4	1	1	2
2.5.1	6	1	1	3
2.7.1	8	1	1	3
2.9.1	10	1	1	3
2.11.1	12	1	1	3
3.3.1	5	1	2	2
3.5.1	7	1	2	4
3.7.1	9	1	2	4
3.9.1	11	1	2	4
3.11.1	13	1	2	4
4.3.1	6	1	3	2
4.5.1	8	1	3	4
5.3.1	7	1	4	2
5.5.1	9	1	4	4
2.5.2	6	2	1	4
2.7.2	8	2	1	5
3.5.2	7	2	2	4
3.7.2	9	2	2	6
3.9.2	11	2	2	6
3.11.2	13	2	2	6

TABLE III.1 (continued)

<u>FORMULA</u>	<u>TOTAL NUMBER OF POINTS USED</u>	<u>HIGHEST ORDER OF CONTINUOUS DERIVATIVE</u>	<u>ORDER OF THE MULTISTEP</u>	<u>LOWEST DEGREE OF POLYNOMIAL USED</u>
4.5.2	8	2	3	4
4.7.2	10	2	3	6
5.5.2	9	2	4	4
2.7.3	8	3	1	6
3.7.3	9	3	2	6
3.9.3	11	3	2	8

TABLE III.2

VALUES OF β AND $\|\alpha F - G\|_E$

FORMULA	$\beta = \min \epsilon_{ij}$	$\ \alpha F - G\ _E$
2.3.1	0	0
2.5.1	$-5.684341886E^{-14}$	$6.200732257E^{-14}$
2.7.1	$-2.842170943E^{-13}$	$4.62490556E^{-13}$
2.9.1	$1.455191523E^{-11}$	$2.105019221E^{-11}$
2.11.1	$-7.275957614E^{-12}$	$1.802631396E^{-11}$
3.3.1	0	0
3.5.1	$3.552713679E^{-15}$	$1.262338714E^{-14}$
3.7.1	$4.618527782E^{-14}$	$1.157744914E^{-13}$
3.9.1	$-5.684341886E^{-13}$	$1.394752431E^{-12}$
3.11.1	$1.818989404E^{-12}$	$5.720967418E^{-12}$
4.3.1	$-1.534772309E^{-12}$	$3.228271377E^{-12}$
4.5.1	$-3.637978807E^{-12}$	$7.530615877E^{-12}$
5.3.1	$-1.47792889E^{-12}$	$1.517869634E^{-12}$
5.5.1	$-3.0127012E^{-12}$	$3.085634286E^{-12}$
2.5.2	$-2.728484105E^{-12}$	$5.074619662E^{-12}$
2.7.2	$-4.365574569E^{-11}$	$7.939614894E^{-11}$
3.5.2	$2.273736754E^{-13}$	$5.039664176E^{-13}$
3.7.2	$-8.185452316E^{-12}$	$1.759273096E^{-11}$
3.9.2	$-2.764863893E^{-10}$	$5.394116113E^{-10}$
3.11.2	$-1.120497473E^{-9}$	$2.531980834E^{-9}$

TABLE III.2 (continued)

<u>FORMULA</u>	<u>$\beta = \min \epsilon_{ij}$</u>	<u>$\ \alpha F - G\ _E$</u>
4.5.2	2.000888344E ⁻¹¹	5.486866514E ⁻¹¹
4.7.2	3.259629011E ⁻⁹	6.949643684E ⁻⁹
5.5.2	8.622009773E ⁻¹⁰	1.9080448E ⁻⁹
2.7.3	7.217749953E ⁻⁹	1.789202523E ⁻⁸
3.7.3	8.731149137E ⁻¹¹	1.996080858E ⁻¹⁰
3.9.3	8.003553376E ⁻¹⁰	2.201444096E ⁻⁹

TABLE III.3

TABLES OF COEFFICIENTS FOR FORMULA 2.5.2

X	$C1$	$C2$	$C3$
$-5.0E^{-01}$	$-1.5612511E^{-17}$	$1.5265567E^{-16}$	$1.00000000E^{00}$
$-4.9E^{-01}$	$8.2879706E^{-04}$	$-6.5984016E^{-03}$	$9.9987214E^{-01}$
$-4.8E^{-01}$	$1.6470860E^{-03}$	$-1.3054090E^{-02}$	$9.9947750E^{-01}$
$-4.7E^{-01}$	$2.4528087E^{-03}$	$-1.9358260E^{-02}$	$9.9880045E^{-01}$
$-4.6E^{-01}$	$3.2440320E^{-03}$	$-2.5502720E^{-02}$	$9.9782656E^{-01}$
$-4.5E^{-01}$	$4.0189453E^{-03}$	$-3.1479883E^{-02}$	$9.9654258E^{-01}$
$-4.4E^{-01}$	$4.7758580E^{-03}$	$-3.7282750E^{-02}$	$9.9493642E^{-01}$
$-4.3E^{-01}$	$5.5131969E^{-03}$	$-4.2904901E^{-02}$	$9.9299713E^{-01}$
$-4.2E^{-01}$	$6.2295040E^{-03}$	$-4.8340480E^{-02}$	$9.9071488E^{-01}$
$-4.1E^{-01}$	$6.9234336E^{-03}$	$-5.3584184E^{-02}$	$9.8808090E^{-01}$
$-4.0E^{-01}$	$7.5937500E^{-03}$	$-5.8631250E^{-02}$	$9.8508750E^{-01}$
$-3.9E^{-01}$	$8.2393252E^{-03}$	$-6.3477442E^{-02}$	$9.8172802E^{-01}$
$-3.8E^{-01}$	$8.8591360E^{-03}$	$-6.8119040E^{-02}$	$9.7799680E^{-01}$
$-3.7E^{-01}$	$9.4522618E^{-03}$	$-7.2552825E^{-02}$	$9.7388918E^{-01}$
$-3.6E^{-01}$	$1.0017882E^{-02}$	$-7.6776070E^{-02}$	$9.6940146E^{-01}$
$-3.5E^{-01}$	$1.0555273E^{-02}$	$-8.0786523E^{-02}$	$9.6453086E^{-01}$
$-3.4E^{-01}$	$1.1063808E^{-02}$	$-8.4582400E^{-02}$	$9.5927552E^{-01}$
$-3.3E^{-01}$	$1.1542950E^{-02}$	$-8.8162367E^{-02}$	$9.5363447E^{-01}$
$-3.2E^{-01}$	$1.1992254E^{-02}$	$-9.1525530E^{-02}$	$9.4760758E^{-01}$
$-3.1E^{-01}$	$1.2411362E^{-02}$	$-9.4671425E^{-02}$	$9.4119558E^{-01}$
$-3.0E^{-01}$	$1.2800000E^{-02}$	$-9.7600000E^{-02}$	$9.3440000E^{-01}$
$-2.9E^{-01}$	$1.3157978E^{-02}$	$-1.0031161E^{-01}$	$9.2722315E^{-01}$
$-2.8E^{-01}$	$1.3485186E^{-02}$	$-1.0280699E^{-01}$	$9.1966810E^{-01}$
$-2.7E^{-01}$	$1.3781590E^{-02}$	$-1.0508727E^{-01}$	$9.1173866E^{-01}$
$-2.6E^{-01}$	$1.4047232E^{-02}$	$-1.0715392E^{-01}$	$9.0343936E^{-01}$
$-2.5E^{-01}$	$1.4282227E^{-02}$	$-1.0900879E^{-01}$	$8.9477539E^{-01}$
$-2.4E^{-01}$	$1.4486758E^{-02}$	$-1.1065405E^{-01}$	$8.8575262E^{-01}$
$-2.3E^{-01}$	$1.4661078E^{-02}$	$-1.1209221E^{-01}$	$8.7637755E^{-01}$
$-2.2E^{-01}$	$1.4805504E^{-02}$	$-1.1332608E^{-01}$	$8.6665728E^{-01}$
$-2.1E^{-01}$	$1.4920415E^{-02}$	$-1.1435879E^{-01}$	$8.5659951E^{-01}$
$-2.0E^{-01}$	$1.5006250E^{-02}$	$-1.1519375E^{-01}$	$8.4621250E^{-01}$
$-1.9E^{-01}$	$1.5063506E^{-02}$	$-1.1583465E^{-01}$	$8.3550503E^{-01}$
$-1.8E^{-01}$	$1.5092736E^{-02}$	$-1.1628544E^{-01}$	$8.2448640E^{-01}$
$-1.7E^{-01}$	$1.5094543E^{-02}$	$-1.1655033E^{-01}$	$8.1316640E^{-01}$
$-1.6E^{-01}$	$1.5069582E^{-02}$	$-1.1663377E^{-01}$	$8.0155526E^{-01}$
$-1.5E^{-01}$	$1.5018555E^{-02}$	$-1.1654043E^{-01}$	$7.8966367E^{-01}$
$-1.4E^{-01}$	$1.4942208E^{-02}$	$-1.1627520E^{-01}$	$7.7750272E^{-01}$
$-1.3E^{-01}$	$1.4841331E^{-02}$	$-1.1584317E^{-01}$	$7.6508388E^{-01}$
$-1.2E^{-01}$	$1.4716754E^{-02}$	$-1.1524963E^{-01}$	$7.5241898E^{-01}$
$-1.1E^{-01}$	$1.4569343E^{-02}$	$-1.1450003E^{-01}$	$7.3952019E^{-01}$
$-1.0E^{-01}$	$1.4400000E^{-02}$	$-1.1360000E^{-01}$	$7.2640000E^{-01}$
$-9.0E^{-02}$	$1.4209660E^{-02}$	$-1.1255531E^{-01}$	$7.1307116E^{-01}$
$-8.0E^{-02}$	$1.3999286E^{-02}$	$-1.1137189E^{-01}$	$6.9954670E^{-01}$
$-7.0E^{-02}$	$1.3769871E^{-02}$	$-1.1005577E^{-01}$	$6.8583988E^{-01}$
$-6.0E^{-02}$	$1.3522432E^{-02}$	$-1.0861312E^{-01}$	$6.7196416E^{-01}$
$-5.0E^{-02}$	$1.3258008E^{-02}$	$-1.0705020E^{-01}$	$6.5793320E^{-01}$
$-4.0E^{-02}$	$1.2977658E^{-02}$	$-1.0537335E^{-01}$	$6.4376082E^{-01}$
$-3.0E^{-02}$	$1.2682459E^{-02}$	$-1.0358901E^{-01}$	$6.2946096E^{-01}$
$-2.0E^{-02}$	$1.2373504E^{-02}$	$-1.0170368E^{-01}$	$6.1504768E^{-01}$
$-1.0E^{-02}$	$1.2051896E^{-02}$	$-9.9723897E^{-02}$	$6.0053513E^{-01}$

TABLE III.3 (continued)

X	$C4$	$C5$	$C6$
$-5.0E^{-01}$	$1.0963452E^{-15}$	$-5.5511151E^{-17}$	$-6.0715322E^{-18}$
$-4.9E^{-01}$	$6.7360319E^{-03}$	$-8.3884969E^{-04}$	$2.8668750E^{-07}$
$-4.8E^{-01}$	$1.3621180E^{-02}$	$-1.6939300E^{-03}$	$2.2540000E^{-06}$
$-4.7E^{-01}$	$2.0670116E^{-02}$	$-2.5725916E^{-03}$	$7.4750625E^{-06}$
$-4.6E^{-01}$	$2.7896320E^{-02}$	$-3.4816000E^{-03}$	$1.7408000E^{-05}$
$-4.5E^{-01}$	$3.5312109E^{-02}$	$-4.4271484E^{-03}$	$3.3398438E^{-05}$
$-4.4E^{-01}$	$4.2928660E^{-02}$	$-5.4148700E^{-03}$	$5.6682000E^{-05}$
$-4.3E^{-01}$	$5.0756033E^{-02}$	$-6.4498503E^{-03}$	$8.8386813E^{-05}$
$-4.2E^{-01}$	$5.8803200E^{-02}$	$-7.5366400E^{-03}$	$1.2953600E^{-04}$
$-4.1E^{-01}$	$6.7078067E^{-02}$	$-8.6792672E^{-03}$	$1.8105019E^{-04}$
$-4.0E^{-01}$	$7.5587500E^{-02}$	$-9.8812500E^{-03}$	$2.4375000E^{-04}$
$-3.9E^{-01}$	$8.4337351E^{-02}$	$-1.1145609E^{-02}$	$3.1835856E^{-04}$
$-3.8E^{-01}$	$9.3332480E^{-02}$	$-1.2474880E^{-02}$	$4.0550400E^{-04}$
$-3.7E^{-01}$	$1.0257678E^{-01}$	$-1.3871126E^{-02}$	$5.0572194E^{-04}$
$-3.6E^{-01}$	$1.1207322E^{-01}$	$-1.5335950E^{-02}$	$6.1945800E^{-04}$
$-3.5E^{-01}$	$1.2182383E^{-01}$	$-1.6870508E^{-02}$	$7.4707031E^{-04}$
$-3.4E^{-01}$	$1.3182976E^{-01}$	$-1.8475520E^{-02}$	$8.8883200E^{-04}$
$-3.3E^{-01}$	$1.4209130E^{-01}$	$-2.0151285E^{-02}$	$1.0449337E^{-03}$
$-3.2E^{-01}$	$1.5260790E^{-01}$	$-2.1897690E^{-02}$	$1.2154860E^{-03}$
$-3.1E^{-01}$	$1.6337819E^{-01}$	$-2.3714227E^{-02}$	$1.4005221E^{-03}$
$-3.0E^{-01}$	$1.7440000E^{-01}$	$-2.5600000E^{-02}$	$1.6000000E^{-03}$
$-2.9E^{-01}$	$1.8567042E^{-01}$	$-2.7553743E^{-02}$	$1.8138054E^{-03}$
$-2.8E^{-01}$	$1.9718578E^{-01}$	$-2.9573830E^{-02}$	$2.0417540E^{-03}$
$-2.7E^{-01}$	$2.0894170E^{-01}$	$-3.1658285E^{-02}$	$2.2835938E^{-03}$
$-2.6E^{-01}$	$2.2093312E^{-01}$	$-3.3804800E^{-02}$	$2.5390080E^{-03}$
$-2.5E^{-01}$	$2.3315430E^{-01}$	$-3.6010742E^{-02}$	$2.8076172E^{-03}$
$-2.4E^{-01}$	$2.4559886E^{-01}$	$-3.8273170E^{-02}$	$3.0889820E^{-03}$
$-2.3E^{-01}$	$2.5825982E^{-01}$	$-4.0588844E^{-02}$	$3.3826056E^{-03}$
$-2.2E^{-01}$	$2.7112960E^{-01}$	$-4.2954240E^{-02}$	$3.6879360E^{-03}$
$-2.1E^{-01}$	$2.8420005E^{-01}$	$-4.5365561E^{-02}$	$4.0043689E^{-03}$
$-2.0E^{-01}$	$2.9746250E^{-01}$	$-4.7818750E^{-02}$	$4.3312500E^{-03}$
$-1.9E^{-01}$	$3.1090774E^{-01}$	$-5.0309503E^{-02}$	$4.6678773E^{-03}$
$-1.8E^{-01}$	$3.2452608E^{-01}$	$-5.2833280E^{-02}$	$5.0135040E^{-03}$
$-1.7E^{-01}$	$3.3830737E^{-01}$	$-5.5385320E^{-02}$	$5.3673407E^{-03}$
$-1.6E^{-01}$	$3.5224102E^{-01}$	$-5.7960650E^{-02}$	$5.7285580E^{-03}$
$-1.5E^{-01}$	$3.6631602E^{-01}$	$-6.0554102E^{-02}$	$6.0962891E^{-03}$
$-1.4E^{-01}$	$3.8052096E^{-01}$	$-6.3160320E^{-02}$	$6.4696320E^{-03}$
$-1.3E^{-01}$	$3.9484409E^{-01}$	$-6.5773778E^{-02}$	$6.8476524E^{-03}$
$-1.2E^{-01}$	$4.0927330E^{-01}$	$-6.8388790E^{-02}$	$7.2293860E^{-03}$
$-1.1E^{-01}$	$4.2379617E^{-01}$	$-7.0999520E^{-02}$	$7.6138408E^{-03}$
$-1.0E^{-01}$	$4.3840000E^{-01}$	$-7.3600000E^{-02}$	$8.0000000E^{-03}$
$-9.0E^{-02}$	$4.5307181E^{-01}$	$-7.6184137E^{-02}$	$8.3868242E^{-03}$
$-8.0E^{-02}$	$4.6779838E^{-01}$	$-7.8745730E^{-02}$	$8.7732540E^{-03}$
$-7.0E^{-02}$	$4.8256629E^{-01}$	$-8.1278479E^{-02}$	$9.1582126E^{-03}$
$-6.0E^{-02}$	$4.9736192E^{-01}$	$-8.3776000E^{-02}$	$9.5406080E^{-03}$
$-5.0E^{-02}$	$5.1217148E^{-01}$	$-8.6231836E^{-02}$	$9.9193359E^{-03}$
$-4.0E^{-02}$	$5.2698106E^{-01}$	$-8.8639470E^{-02}$	$1.0293282E^{-02}$
$-3.0E^{-02}$	$5.4177661E^{-01}$	$-9.0992338E^{-02}$	$1.0661324E^{-02}$
$-2.0E^{-02}$	$5.5654400E^{-01}$	$-9.3283840E^{-02}$	$1.1022336E^{-02}$
$-1.0E^{-02}$	$5.7126904E^{-01}$	$-9.5507355E^{-02}$	$1.1375188E^{-02}$

TABLE III.3 (continued)

X	$C1$	$C2$	$C3$
$-5.2E^{-18}$	$1.1718750E^{-02}$	$-9.7656250E^{-02}$	$5.8593750E^{-01}$
$1.0E^{-02}$	$1.1375188E^{-02}$	$-9.5507355E^{-02}$	$5.7126904E^{-01}$
$2.0E^{-02}$	$1.1022336E^{-02}$	$-9.3283840E^{-02}$	$5.5654400E^{-01}$
$3.0E^{-02}$	$1.0661324E^{-02}$	$-9.0992338E^{-02}$	$5.4177661E^{-01}$
$4.0E^{-02}$	$1.0293282E^{-02}$	$-8.8639470E^{-02}$	$5.2698106E^{-01}$
$5.0E^{-02}$	$9.9193359E^{-03}$	$-8.6231836E^{-02}$	$5.1217148E^{-01}$
$6.0E^{-02}$	$9.5406080E^{-03}$	$-8.3776000E^{-02}$	$4.9736192E^{-01}$
$7.0E^{-02}$	$9.1582126E^{-03}$	$-8.1278479E^{-02}$	$4.8256629E^{-01}$
$8.0E^{-02}$	$8.7732540E^{-03}$	$-7.8745730E^{-02}$	$4.6779838E^{-01}$
$9.0E^{-02}$	$8.3868242E^{-03}$	$-7.6184137E^{-02}$	$4.5307181E^{-01}$
$1.0E^{-01}$	$8.0000000E^{-03}$	$-7.3600000E^{-02}$	$4.3840000E^{-01}$
$1.1E^{-01}$	$7.6138408E^{-03}$	$-7.0999520E^{-02}$	$4.2379617E^{-01}$
$1.2E^{-01}$	$7.2293860E^{-03}$	$-6.8388790E^{-02}$	$4.0927330E^{-01}$
$1.3E^{-01}$	$6.8476524E^{-03}$	$-6.5773778E^{-02}$	$3.9484409E^{-01}$
$1.4E^{-01}$	$6.4696320E^{-03}$	$-6.3160320E^{-02}$	$3.8052096E^{-01}$
$1.5E^{-01}$	$6.0962891E^{-03}$	$-6.0554102E^{-02}$	$3.6631602E^{-01}$
$1.6E^{-01}$	$5.7285580E^{-03}$	$-5.7960650E^{-02}$	$3.5224102E^{-01}$
$1.7E^{-01}$	$5.3673407E^{-03}$	$-5.5385320E^{-02}$	$3.3830737E^{-01}$
$1.8E^{-01}$	$5.0135040E^{-03}$	$-5.2833280E^{-02}$	$3.2452608E^{-01}$
$1.9E^{-01}$	$4.6678773E^{-03}$	$-5.0309503E^{-02}$	$3.1090774E^{-01}$
$2.0E^{-01}$	$4.3312500E^{-03}$	$-4.7818750E^{-02}$	$2.9746250E^{-01}$
$2.1E^{-01}$	$4.0043689E^{-03}$	$-4.5365561E^{-02}$	$2.8420005E^{-01}$
$2.2E^{-01}$	$3.6879360E^{-03}$	$-4.2954240E^{-02}$	$2.7112960E^{-01}$
$2.3E^{-01}$	$3.3826056E^{-03}$	$-4.0588844E^{-02}$	$2.5825982E^{-01}$
$2.4E^{-01}$	$3.0889820E^{-03}$	$-3.8273170E^{-02}$	$2.4559886E^{-01}$
$2.5E^{-01}$	$2.8076172E^{-03}$	$-3.6010742E^{-02}$	$2.3315430E^{-01}$
$2.6E^{-01}$	$2.5390080E^{-03}$	$-3.3804800E^{-02}$	$2.2093312E^{-01}$
$2.7E^{-01}$	$2.2835938E^{-03}$	$-3.1658285E^{-02}$	$2.0894170E^{-01}$
$2.8E^{-01}$	$2.0417540E^{-03}$	$-2.9573830E^{-02}$	$1.9718578E^{-01}$
$2.9E^{-01}$	$1.8138054E^{-03}$	$-2.7553743E^{-02}$	$1.8567042E^{-01}$
$3.0E^{-01}$	$1.6000000E^{-03}$	$-2.5600000E^{-02}$	$1.7440000E^{-01}$
$3.1E^{-01}$	$1.4005221E^{-03}$	$-2.3714227E^{-02}$	$1.6337819E^{-01}$
$3.2E^{-01}$	$1.2154860E^{-03}$	$-2.1897690E^{-02}$	$1.5260790E^{-01}$
$3.3E^{-01}$	$1.0449337E^{-03}$	$-2.0151285E^{-02}$	$1.4209130E^{-01}$
$3.4E^{-01}$	$8.8883200E^{-04}$	$-1.8475520E^{-02}$	$1.3182976E^{-01}$
$3.5E^{-01}$	$7.4707031E^{-04}$	$-1.6870508E^{-02}$	$1.2182383E^{-01}$
$3.6E^{-01}$	$6.1945800E^{-04}$	$-1.5335950E^{-02}$	$1.1207322E^{-01}$
$3.7E^{-01}$	$5.0572194E^{-04}$	$-1.3871126E^{-02}$	$1.0257678E^{-01}$
$3.8E^{-01}$	$4.0550400E^{-04}$	$-1.2474880E^{-02}$	$9.3332480E^{-02}$
$3.9E^{-01}$	$3.1835856E^{-04}$	$-1.1145609E^{-02}$	$8.4337351E^{-02}$
$4.0E^{-01}$	$2.4375000E^{-04}$	$-9.8812500E^{-03}$	$7.5587500E^{-02}$
$4.1E^{-01}$	$1.8105019E^{-04}$	$-8.6792672E^{-03}$	$6.7078067E^{-02}$
$4.2E^{-01}$	$1.2953600E^{-04}$	$-7.5366400E^{-03}$	$5.8803200E^{-02}$
$4.3E^{-01}$	$8.8386813E^{-05}$	$-6.4498503E^{-03}$	$5.0756033E^{-02}$
$4.4E^{-01}$	$5.6682000E^{-05}$	$-5.4148700E^{-03}$	$4.2928660E^{-02}$
$4.5E^{-01}$	$3.3398438E^{-05}$	$-4.4271484E^{-03}$	$3.5312109E^{-02}$
$4.6E^{-01}$	$1.7408000E^{-05}$	$-3.4816000E^{-03}$	$2.7896320E^{-02}$
$4.7E^{-01}$	$7.4750625E^{-06}$	$-2.5725916E^{-03}$	$2.0670116E^{-02}$
$4.8E^{-01}$	$2.2540000E^{-06}$	$-1.6939300E^{-03}$	$1.3621180E^{-02}$
$4.9E^{-01}$	$2.8668750E^{-07}$	$-8.3884969E^{-04}$	$6.7360319E^{-03}$
$5.0E^{-01}$	$-3.4694470E^{-18}$	$2.7755576E^{-17}$	$1.4571677E^{-15}$

TABLE III.3 (continued)

X	$C4$	$C5$	$C6$
$-5.2E^{-18}$	$5.8593750E^{-01}$	$-9.7656250E^{-02}$	$1.1718750E^{-02}$
$1.0E^{-02}$	$6.0053513E^{-01}$	$-9.9723897E^{-02}$	$1.2051896E^{-02}$
$2.0E^{-02}$	$6.1504768E^{-01}$	$-1.0170368E^{-01}$	$1.2373504E^{-02}$
$3.0E^{-02}$	$6.2946096E^{-01}$	$-1.0358901E^{-01}$	$1.2682459E^{-02}$
$4.0E^{-02}$	$6.4376082E^{-01}$	$-1.0537335E^{-01}$	$1.2977658E^{-02}$
$5.0E^{-02}$	$6.5793320E^{-01}$	$-1.0705020E^{-01}$	$1.3258008E^{-02}$
$6.0E^{-02}$	$6.7196416E^{-01}$	$-1.0861312E^{-01}$	$1.3522432E^{-02}$
$7.0E^{-02}$	$6.8583988E^{-01}$	$-1.1005577E^{-01}$	$1.3769871E^{-02}$
$8.0E^{-02}$	$6.9954670E^{-01}$	$-1.1137189E^{-01}$	$1.3999286E^{-02}$
$9.0E^{-02}$	$7.1307116E^{-01}$	$-1.1255531E^{-01}$	$1.4209660E^{-02}$
$1.0E^{-01}$	$7.2640000E^{-01}$	$-1.1360000E^{-01}$	$1.4400000E^{-02}$
$1.1E^{-01}$	$7.3952019E^{-01}$	$-1.1450003E^{-01}$	$1.4569343E^{-02}$
$1.2E^{-01}$	$7.5241898E^{-01}$	$-1.1524963E^{-01}$	$1.4716754E^{-02}$
$1.3E^{-01}$	$7.6508388E^{-01}$	$-1.1584317E^{-01}$	$1.4841331E^{-02}$
$1.4E^{-01}$	$7.7750272E^{-01}$	$-1.1627520E^{-01}$	$1.4942208E^{-02}$
$1.5E^{-01}$	$7.8966367E^{-01}$	$-1.1654043E^{-01}$	$1.5018555E^{-02}$
$1.6E^{-01}$	$8.0155526E^{-01}$	$-1.1663377E^{-01}$	$1.5069582E^{-02}$
$1.7E^{-01}$	$8.1316640E^{-01}$	$-1.1655033E^{-01}$	$1.5094543E^{-02}$
$1.8E^{-01}$	$8.2448640E^{-01}$	$-1.1628544E^{-01}$	$1.5092736E^{-02}$
$1.9E^{-01}$	$8.3550503E^{-01}$	$-1.1583465E^{-01}$	$1.5063506E^{-02}$
$2.0E^{-01}$	$8.4621250E^{-01}$	$-1.1519375E^{-01}$	$1.5006250E^{-02}$
$2.1E^{-01}$	$8.5659951E^{-01}$	$-1.1435879E^{-01}$	$1.4920415E^{-02}$
$2.2E^{-01}$	$8.6665728E^{-01}$	$-1.1332608E^{-01}$	$1.4805504E^{-02}$
$2.3E^{-01}$	$8.7637755E^{-01}$	$-1.1209221E^{-01}$	$1.4661078E^{-02}$
$2.4E^{-01}$	$8.8575262E^{-01}$	$-1.1065405E^{-01}$	$1.4486758E^{-02}$
$2.5E^{-01}$	$8.9477539E^{-01}$	$-1.0900879E^{-01}$	$1.4282227E^{-02}$
$2.6E^{-01}$	$9.0343936E^{-01}$	$-1.0715392E^{-01}$	$1.4047232E^{-02}$
$2.7E^{-01}$	$9.1173866E^{-01}$	$-1.0508727E^{-01}$	$1.3781590E^{-02}$
$2.8E^{-01}$	$9.1966810E^{-01}$	$-1.0280699E^{-01}$	$1.3485186E^{-02}$
$2.9E^{-01}$	$9.2722315E^{-01}$	$-1.0031161E^{-01}$	$1.3157978E^{-02}$
$3.0E^{-01}$	$9.3440000E^{-01}$	$-9.7600000E^{-02}$	$1.2800000E^{-02}$
$3.1E^{-01}$	$9.4119558E^{-01}$	$-9.4671425E^{-02}$	$1.2411362E^{-02}$
$3.2E^{-01}$	$9.4760758E^{-01}$	$-9.1525530E^{-02}$	$1.1992254E^{-02}$
$3.3E^{-01}$	$9.5363447E^{-01}$	$-8.8162367E^{-02}$	$1.1542950E^{-02}$
$3.4E^{-01}$	$9.5927552E^{-01}$	$-8.4582400E^{-02}$	$1.1063808E^{-02}$
$3.5E^{-01}$	$9.6453086E^{-01}$	$-8.0786523E^{-02}$	$1.0555273E^{-02}$
$3.6E^{-01}$	$9.6940146E^{-01}$	$-7.6776070E^{-02}$	$1.0017882E^{-02}$
$3.7E^{-01}$	$9.7388918E^{-01}$	$-7.2552825E^{-02}$	$9.4522618E^{-03}$
$3.8E^{-01}$	$9.7799680E^{-01}$	$-6.8119040E^{-02}$	$8.8591360E^{-03}$
$3.9E^{-01}$	$9.8172802E^{-01}$	$-6.3477442E^{-02}$	$8.2393252E^{-03}$
$4.0E^{-01}$	$9.8508750E^{-01}$	$-5.8631250E^{-02}$	$7.5937500E^{-03}$
$4.1E^{-01}$	$9.8808090E^{-01}$	$-5.3584184E^{-02}$	$6.9234336E^{-03}$
$4.2E^{-01}$	$9.9071488E^{-01}$	$-4.8340480E^{-02}$	$6.2295040E^{-03}$
$4.3E^{-01}$	$9.9299713E^{-01}$	$-4.2904901E^{-02}$	$5.5131969E^{-03}$
$4.4E^{-01}$	$9.9493642E^{-01}$	$-3.7282750E^{-02}$	$4.7758580E^{-03}$
$4.5E^{-01}$	$9.9654258E^{-01}$	$-3.1479883E^{-02}$	$4.0189453E^{-03}$
$4.6E^{-01}$	$9.9782656E^{-01}$	$-2.5502720E^{-02}$	$3.2440320E^{-03}$
$4.7E^{-01}$	$9.9880045E^{-01}$	$-1.9358260E^{-02}$	$2.4528087E^{-03}$
$4.8E^{-01}$	$9.9947750E^{-01}$	$-1.3054090E^{-02}$	$1.6470860E^{-03}$
$4.9E^{-01}$	$9.9987214E^{-01}$	$-6.5984016E^{-03}$	$8.2879706E^{-04}$
$5.0E^{-01}$	$1.0000000E00$	$4.4408921E^{-16}$	$-1.9949320E^{-17}$

APPENDIX IV

NUMERICAL RESULTS OF THE TEST FUNCTIONS AND PROCEDURE

In this appendix three tables are given. Table IV.1 gives the number of multiplications required in the application of each coefficient matrix F in (3.53) by the procedure given in section 5.2 of Chapter V. The number of multiplications required for the set of coefficients of H corresponds to that required in calculating all the coefficients e_j of (5.4) for each coefficient matrix F of (3.53). The number of multiplications in one evaluation of H is that required in evaluating a polynomial of the form (5.6). In Tables IV.2 and IV.3 are given the maximum relative errors for (5.1) and (5.2) respectively. The two last tables were obtained using the functions $\text{APPLIC}\Delta\text{CPIF}$, POLY , CPI , and INVERSE . The results in Table IV.2 are for a constant interval i.e. $[1.025,2]$, while those in Table IV.3 are for the intervals $[10.025,11]$ and $[25.025,26]$.

TABLE IV.1

NUMBER OF MULTIPLICATIONS REQUIRED

<u>FORMULA</u>	<u>THE SET OF COEFFICIENTS OF H</u>	<u>ONE EVALUATION OF H</u>
2.3.1	8	3
2.5.1	12	3
2.7.1	16	3
2.9.1	20	3
2.11.1	24	3
3.3.1	11	4
3.5.1	15	4
3.7.1	19	4
3.9.1	23	4
3.11.1	27	4
4.3.1	18	5
4.5.1	24	5
5.3.1	22	6
5.5.1	28	6
2.5.2	18	5
2.7.2	24	5
3.5.2	22	6
3.7.2	28	6
3.9.2	34	6
3.11.2	40	6
4.5.2	32	7
4.7.2	40	7

TABLE IV.1 (continued)

<u>FORMULA</u>	<u>THE SET OF COEFFICIENTS OF H</u>	<u>ONE EVALUATION OF H</u>
5.5.2	37	8
2.7.3	32	7
3.7.3	37	8
3.9.3	45	8

TABLE IV.2

THE LARGEST RELATIVE ERROR FOR x^i , $i = 6, 7, 10$

<u>FORMULA</u>	<u>x^6</u>	<u>x^7</u>	<u>x^{10}</u>
2.3.1	4.96431E ⁻³	1.03283E ⁻²	4.56391E ⁻²
2.5.1	3.29647E ⁻⁴	7.59429E ⁻⁴	-3.36421E ⁻³
2.7.1	2.96472E ⁻⁴	6.92587E ⁻⁴	4.34825E ⁻³
2.9.1	2.96472E ⁻⁴	6.92587E ⁻⁴	4.18379E ⁻³
2.11.1	2.96472E ⁻⁴	6.92587E ⁻⁴	4.18507E ⁻³
3.3.1	9.12287E ⁻³	1.65640E ⁻²	6.60124E ⁻²
3.5.1	-1.20968E ⁻⁴	-4.40017E ⁻⁴	-6.23479E ⁻³
3.7.1	7.28776E ⁻⁵	2.27503E ⁻⁴	2.93142E ⁻³
3.9.1	7.28776E ⁻⁵	2.22446E ⁻⁴	2.27444E ⁻³
3.11.1	7.28776E ⁻⁵	2.22446E ⁻⁴	2.28882E ⁻³
4.3.1	1.08089E ⁻²	2.00979E ⁻²	9.59634E ⁻²
4.5.1	-2.42844E ⁻⁴	-8.99391E ⁻⁴	-1.30269E ⁻²
5.3.1	6.85474E ⁻³	1.24611E ⁻²	4.89673E ⁻²
5.5.1	-1.31759E ⁻⁴	-4.69175E ⁻⁴	-6.28779E ⁻³
2.5.2	-1.23017E ⁻⁴	-5.00746E ⁻⁴	-7.54654E ⁻³
2.7.2	-8.77915E ⁻⁸	-4.90022E ⁻⁶	4.65282E ⁻⁴
3.5.2	-3.00247E ⁻⁴	-1.07975E ⁻³	-1.33627E ⁻²
3.7.2	2.58471E ⁻¹⁵	-2.10842E ⁻⁵	9.01192E ⁻⁴
3.9.2	2.71395E ⁻¹⁵	-1.95108E ⁻⁷	-4.43514E ⁻⁵
3.11.2	-2.94344E ⁻¹⁵	-1.95108E ⁻⁷	-2.03496E ⁻⁵
4.5.2	-4.79520E ⁻⁴	-1.82021E ⁻³	-2.74970E ⁻²
4.7.2	-2.72116E ⁻¹⁵	1.50320E ⁻⁵	2.33937E ⁻³

TABLE IV.2 (continued)

<u>FORMULA</u>	<u>x⁶</u>	<u>x⁷</u>	<u>x¹⁰</u>
5.5.2	$-2.52548E^{-4}$	$-9.16153E^{-4}$	$-1.29137E^{-2}$
2.7.3	$-1.44329E^{-15}$	$-5.50463E^{-6}$	$5.56570E^{-4}$
3.7.3	$4.45544E^{-15}$	$-2.37197E^{-5}$	$1.17684E^{-3}$
3.9.3	$3.70028E^{-15}$	$-3.58047E^{-15}$	$-2.53198E^{-5}$

Note: Interval used was [1.025,2] in increments of .025

TABLE IV.3

THE LARGEST RELATIVE ERROR FOR e^x

FORMULA	INTERVAL [10.025,11]	INTERVAL [25.025,26]
2. 3 .1	$^{-2.22957E^{-7}}$	$^{-1.00527E^{-8}}$
2. 5 .1	$^{-2.46006E^{-9}}$	$^{-4.73603E^{-11}}$
2. 7 .1	$^{-2.50947E^{-9}}$	$^{-4.75101E^{-11}}$
2. 9 .1	$^{-2.50920E^{-9}}$	$^{-4.75101E^{-11}}$
2.11 .1	$^{-2.50920E^{-9}}$	$^{-4.75101E^{-11}}$
3. 3 .1	$^{-4.20991E^{-7}}$	$^{-1.96316E^{-8}}$
3. 5 .1	$4.47228E^{-10}$	$3.28966E^{-12}$
3. 7 .1	$^{-2.14235E^{-10}}$	$^{-1.67370E^{-12}}$
3. 9 .1	$^{-2.12127E^{-10}}$	$^{-1.67109E^{-12}}$
3.11 .1	$^{-2.12143E^{-10}}$	$^{-1.67102E^{-12}}$
4. 3 .1	$^{-2.96954E^{-7}}$	$^{-1.34385E^{-8}}$
4. 5 .1	$4.40824E^{-10}$	$3.21082E^{-12}$
5. 3 .1	$^{-3.29027E^{-7}}$	$^{-1.53183E^{-8}}$
5. 5 .1	$4.88252E^{-10}$	$3.65441E^{-12}$
2. 5 .2	$4.88991E^{-10}$	$3.55196E^{-12}$
2. 7 .2	$^{-1.91871E^{-12}}$	$2.81353E^{-15}$
3. 5 .2	$9.47390E^{-10}$	$7.13548E^{-12}$
3. 7 .2	$^{-3.69734E^{-12}}$	$5.35420E^{-15}$
3. 9 .2	$1.04149E^{-13}$	$1.64445E^{-15}$
3.11 .2	$7.71296E^{-14}$	$1.58609E^{-15}$
4. 5 .2	$7.90188E^{-10}$	$5.72110E^{-12}$
4. 7 .2	$^{-3.21132E^{-12}}$	$4.10618E^{-15}$
5. 5 .2	$8.64365E^{-10}$	$6.43914E^{-12}$

TABLE IV.3 (continued)

<u>FORMULA</u>	<u>INTERVAL [10.025,11]</u>	<u>INTERVAL [25.025,26]</u>
2. 7 .3	$^{-2.39110E^{-12}}$	$^{-3.10131E^{-15}}$
3. 7 .3	$^{-4.21270E^{-12}}$	$5.97381E^{-15}$
3. 9 .3	$3.34744E^{-14}$	$3.47573E^{-15}$

Note: Increments used were .025

